COMPLEX ANALYSIS

The study of complex numbers with algebra is complex analysis

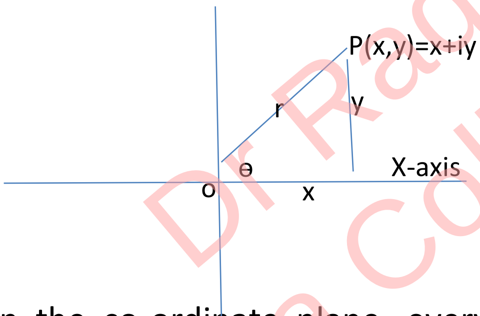
Complex number: Let a and b are any two real numbers, the number of the form a+ib, where i=$\sqrt{-1}$ is called called a complex number. a and b are respectively real and imaginary parts of complex number. Ex: 2+i3, 3-i$\sqrt{2}$, $\frac{1}{2}+\frac{i}{3}$, etc.

Complex Variable: Let x and y be any two real variables, then z=x+iy is called a complex variable.

Here x is called real part of Z denoted by Re(z) and y is imaginary part of z denoted by Im(z)

If z=x+iy is a complex variable, then its conjugate is $\overline{z}$ = x-iy,

Representation of a complex number



In the co-ordinate plane, every point is a pair of real numbers which is a complex number. Every point on X-axis whose y co-ordinate is 0, thus a complex number on X-axis has imaginary part zero called real axis. Similarly, on Y-axis has real part zero called imaginary axis.

We have, From the above diagram, tanɵ=$\frac{y}{x}$, sinɵ=$\frac{y}{r}$, cosɵ=$\frac{x}{r}$, where r=$\sqrt{x^2+y^2}$ is magnitude or modulus of the complex number gives the length of the complex number from the origin and ɵ=$tan^{-1}\left(\frac{y}{x}\right)$is called amplitude or argument of complex number gives the amount of rotation from the initial position.

z=x+iy = r(cosɵ+isinɵ) = r$e^{i\theta}$

Cartesian form, polar form, exponential form

Algebra of Complex Numbers:

Addition, subtraction, multiplication and division(with denominator non-zero) of complex numbers is a complex number.

Simple Problems:

Find modulus, amplitude and express in polar form of:

1) $1+i\sqrt{3}$

Modulus=r = $\sqrt{1^2+\sqrt{3}^2}$=2, amplitude = $\theta=tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

Polar form is $\sqrt{3}+i$ = r(cosθ+isinθ)=$2(\cos\frac{\pi}{3}+isin\frac{\pi}{3})$

2) $1-i\sqrt{3}$

Modulus=r = $\sqrt{1^2+(-\sqrt{3})^2}$=2, amplitude = $\theta=tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)=tan^{-1}(-\sqrt{3})=\frac{-\pi}{3}$

Polar form is $1-i\sqrt{3}$ = r(cosθ+isinθ)=$2(\cos(\frac{-\pi}{3})+isin(\frac{-\pi}{3}))$= $2(\cos\frac{\pi}{3}-isin\frac{\pi}{3})$

3) $-1+i\sqrt{3}$

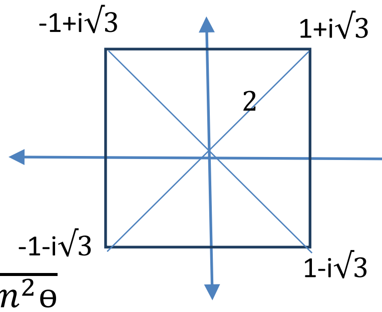
Modulus=r = $\sqrt{(-1)^2+(\sqrt{3})^2}$=2, amplitude = $\theta=tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Polar form is $-1+i\sqrt{3}$ = r(cosθ+isinθ)=$2(\cos\frac{2\pi}{3}+isin\frac{2\pi}{3})$

4) $-1-i\sqrt{3}$

Modulus=r = $\sqrt{(-1)^2+(-\sqrt{3})^2}$=2, amplitude = $\theta=tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$

Polar form is $-1-i\sqrt{3}$ = r(cosθ+isinθ)=$2(\cos\frac{4\pi}{3}+isin\frac{4\pi}{3})$



5) 1+cosθ +i sinθ

Modulus=r = $\sqrt{(1+cos\theta)^2+(sin\Theta)^2)}$=$\sqrt{1+2cos\theta+cos^2\theta+sin^2\theta}$

$$=\sqrt{2+2cos\theta}$$

$$= \sqrt{2(1 + cos\theta)} = \sqrt{4cos^2\frac{\theta}{2}} = 2\cos\frac{\theta}{2},$$

amplitude = $\alpha = tan^{-1}\left(\frac{sin\theta}{1+cos\theta}\right) = tan^{-1}\left(\frac{2sin\frac{\theta}{2}cos\frac{\theta}{2}}{2cos^2\frac{\theta}{2}}\right) = \frac{\theta}{2}$

Polar form is $1+\cos\theta + i\sin\theta = r(\cos\alpha + i\sin\alpha) = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + isin\frac{\theta}{2}\right)$

Properties of Complex Numbers

Let z_1, z_2 be any two complex numbers, then

1. $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$

2. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, where $z_2 \neq 0$

3. $|z_1 \, z_2| = |z_1||z_2|$

4. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $z_2 \neq 0$

5. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

6. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

7. $|z_1 + z_2| \leq |z_1| + |z_2|$

8. $|z_1 - z_2| \geq |z_1| - |z_2|$, equality holds only if $z_1 = z_2$

9. $z + \bar{z} = 2\text{Re}(z)$

10. $z - \bar{z} = 2\text{Im}(z)$

11. $z \, \bar{z} = |z|^2$

Euler's Formula

Using Taylor's Theorem, we have

$$f(x) = 1 + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x4}{4!} f''''(0) + -----$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + -----$$

replace x by $i\theta$, we have

$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} -----$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + -----\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} -----\right)$$

$$\boldsymbol{e^{i\theta} = \cos\theta + i \sin\theta}$$, where $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + ----$ and $\sin\theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} ----$

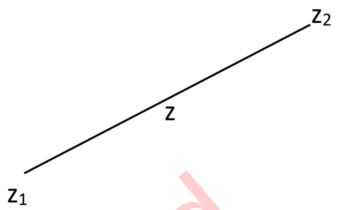
Euler's Formula

we have, **cos(ix)=coshx** = $\frac{e^{x}+e^{-x}}{2}$ and **sin(ix)=i sinhx** =i $\left(\frac{e^{x}-e^{-x}}{2}\right)$

Equation of straight line in complex form

Equation of straight line passing through two different points z_1 and z_2

Proof: let z_1 and z_2 be any two complex points on a straight line. let z be any arbitrary point on the line.



since z_1, z and z_2 are collinear, then

$$\arg\left(\frac{z-z_1}{z_2-z_1}\right)=0$$

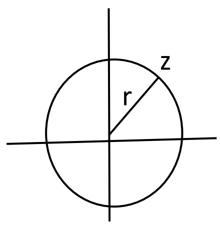
$$\left(\frac{z-z_1}{z_2-z_1}\right)=\overline{\left(\frac{z-z_1}{z_2-z_1}\right)}$$

$$\left(\frac{z-z_1}{z_2-z_1}\right)=\frac{\overline{z-z_1}}{\overline{z_2-z_1}}$$

$$(z-z_1)(\overline{z_2-z_1})=(\overline{z-z_1})(z_2-z_1)$$

$$(z-z_1)(\bar{z_2}-\bar{z_1})=(\bar{z}-\bar{z_1})(z_2-z_1)$$

Equation of circle

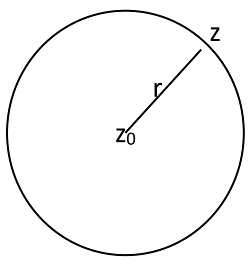


I. Equation of circle having center at origin,

radius be r and z be any point on the circle is

$|z-0|$=r or $|z|$ =r or z =r $e^{i\theta}$ because $|e^{i\theta}|$=1

II. Equation of circle having center at z_0 and radius r



$|z-z_0|$=r or z - z_0= r $e^{i\theta}$, because $|e^{i\theta}|$=1

$z=z_0+ r e^{i\theta}$

Problems:

1. Find the locus of the point z satisfying $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}$

Soln: $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{3}$

Note: $\arg(x+iy)=\tan^{-1}\left(\frac{y}{x}\right)$

$$\arg\left(\frac{x+iy-1}{x+iy+1}\right)=\frac{\pi}{3} \Longrightarrow \arg\left(\frac{x-1+iy}{x+1+iy}\right)=\frac{\pi}{3}$$

$\arg(x-1+iy) - \arg(x+1+iy)=\frac{\pi}{3}$

$\tan^{-1}\left(\frac{y}{x-1}\right)$- $\tan^{-1}\left(\frac{y}{x+1}\right)$=$\frac{\pi}{3}$

$\tan^{-1}\left(\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y}{x-1}\cdot\frac{y}{x+1}}\right)$=$\frac{\pi}{3}$

$\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y}{x-1}\cdot\frac{y}{x+1}}$=tan$\frac{\pi}{3}$

$\frac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}$ =$\sqrt{3}$

$\frac{2y}{(x-1)(x+1)+y^2}$ =$\sqrt{3}$ $\Longrightarrow$ 2y=$\sqrt{3}((x-1)(x+1)+y^2)$

2y = $\sqrt{3}(x^2-1+y^2)$ $\Longrightarrow$$\sqrt{3}(x^2+y^2-1)=2y$

$x^2+y^2-\frac{2}{\sqrt{3}}y$ -1=0 is a circle whose centre=(0, $\frac{1}{\sqrt{3}}$), radius=$\sqrt{(\frac{1}{\sqrt{3}})^2+1}=\frac{2}{\sqrt{3}}$

Note: $x^2 + y^2 + 2gx + 2fy + c=0$ is a circle, centre=(-g,-f), radius=$\sqrt{g^2+f^2-c}$

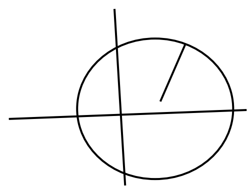
2. Find the locus of the point z satisfying $|z-1|$ ≥2

soln: $|z-1| \geq 2$

$|x+iy-1| \geq 2$

$|x-1+iy| \geq 2$

$|x-1+iy|^2 \geq 4$



$(x-1)^2 + y^2 \geq 4$ is the boundary points and out side the circle whose centre=(1,0) and radius=2

3. If $\left(\frac{z-i}{z-1}\right)$ is purely imaginary, then show that its locus is a circle.

Soln: $\left(\frac{z-i}{z-1}\right)$ is purely imaginary

real part is 0

$\left(\frac{z-i}{z-1}\right)$=$\left(\frac{x+iy-i}{x+iy-1}\right)$=$\frac{x+i(y-1)}{x-1+iy}$

multiply and divide x-1-iy

$=\frac{x+i(y-1)}{x-1+iy}\text{x}\frac{x-1-iy}{x-1-iy}$

$=\frac{x^2-x+y^2-y+i[(x-1)(y-1)-xy]}{(x-1)^2+y^2}=\frac{x^2-x+y^2-y]}{(x-1)^2+y^2}+i\ \frac{[(x-1)(y-1)-xy]}{(x-1)^2+y^2}$

Real part =0

$\frac{x^2+y^2-x-y}{(x-1)^2+y^2}=0$

$x^2+y^2-x-y=0$ is a circle whose centre=$\left(\frac{1}{2},\frac{1}{2}\right)$ and radius=$\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$ $=\frac{1}{\sqrt{2}}$

4. Show that arg$\left(\frac{\bar{z}}{z}\right)=\frac{\pi}{2}$ is a line through origin.

soln: arg$\left(\frac{\bar{z}}{z}\right)=\frac{\pi}{2}$

arg ($\bar{z}$)- arg(z) = $\frac{\pi}{2}$

arg(x-iy) - arg(x+iy)= $\frac{\pi}{2}$

$$\tan^{-1}\left(\frac{-y}{x}\right)-\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}$$

$$-\tan^{-1}\left(\frac{y}{x}\right)-\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}$$

$$-2\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{y}{x}\right)=\frac{-\pi}{4}$$

$$\frac{y}{x}=\tan\left(\frac{-\pi}{4}\right)$$

$\frac{y}{x} = -1$

x+y=0, represents a straight line through origin.

5. Find the locus of the point z satisfying $|z+i| \leq 3$

soln:

$|\mathbf{z} + \mathbf{i}| \leq 3$

$|\mathbf{x} + \mathbf{iy} + \mathbf{i}| \leq 3$

$|\mathbf{x} + \mathbf{i}(\mathbf{y} + \mathbf{1})| \leq 3$

$|\mathbf{x} + \mathbf{i}(\mathbf{y} + \mathbf{1})|^2 \leq 9$

$x^2 + (y+1)^2 \leq 9$

$x^2 + y^2 + 2y - 8 \leq 0$ is interior and boundary points of a circle whose centre=(0,-1) and radius = $\sqrt{(-1)^2 + 8} = 3$

6. Find the locus of the point z satisfying $|z-1| + |z+1| \leq 4$

Soln: $|z-1| + |z+1| \leq 4$

on squaring

$(|z-1| + |z+1|)^2 \leq 16$

$|z-1|^2 + |z+1|^2 + 2(|z-1|.|z+1|) \leq 16$

$|z|^2 - 2z + 1 + |z|^2 + 2z + 1 + 2|z^2 - 1| \leq 16$

$2|z|^2 + 2|z^2 - 1| + 2 \leq 16$

$|z|^2 + |z^2 - 1| + 1 \leq 8$

$x^2 + y^2 + |(x+iy)^2 - 1| \leq 7$

$|x^2 - y^2 + i2xy - 1| \leq 7 - x^2 - y^2$

$|x^2 - y^2 + i2xy - 1| \le 7 - x^2 - y^2$

on squaring

$|x^2 - y^2 + i2xy - 1|^2 \le (7 - x^2 - y^2)^2$

$|x^2 - y^2 - 1 + i2xy|^2 \le 49 + x^4 + y^4 - 14x^2 + 2x^2y^2 - 14y^2$

$(x^2 - y^2 - 1)^2 + 4x^2y^2 \le x^4 + y^4 - 14x^2 + 2x^2y^2 - 14y^2 + 49$

$x^4 + y^4 + 1 - 2x^2y^2 + 2y^2 - 2x^2 + 4x^2y^2 - x^4 - y^4 + 14x^2 - 2x^2y^2 + 14y^2 - 49 \le 0$

$12x^2 + 16y^2 - 48 \le 0$

$12x^2 + 16y^2 \le 48$

$\frac{x^2}{4} + \frac{y^2}{3} \le 1$

represents boundary and interior pointa of ellipse whose centre=(0,0),

major axis=4, minor axis=2$\sqrt{6}$

Assignment:

1. Show that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ represents a circle.

2. Show that $|z - 1 + 2i| = 4$ represent a circle and its position.

Complex Function:

Let z be any complex variable. For each value of z=x+iy a comlex variable there correspongs to unique value of f(z)=u+iv is called a complex fuction, where u=u(x,y) and v=v(x,y) be the real valued functions.

Ex: 1. $f(z)=z^2$

$f(z)=(x+iy)^2=x^2+(iy)^2+i2xy$

$=x^2-y^2+i(2xy)$

$\Longrightarrow u(x,y)= x^2-y^2,\ v(x,y)= 2xy$

2. f(z)=sinz

f(z)=sin(x+iy)

=sinx cos(iy) + cosx sin(iy)

=sinx coshy + cosx (isinhy)   since, **cos(iy)=coshy and sin(iy)= isinhy**

= sinx coshy +i cosx sinhy

$\Longrightarrow$ u(x,y)= sinx coshy, v(x,y)= cosx sinhy

3. $f(z)=e^z$

$f(z)=e^z$

$= e^{x+iy}$

$=e^x e^{iy}$

$=e^x(\cos y + i\sin y)$,   where, $e^{iy}= \cos y + i\sin y$

$=e^x \cos y + i(e^x \cos y)$

$u(x,y)= e^x \cos y,\ v(x,y)= e^x \sin y$

Limit of a complex function:

Let W=f(z) be any function of z defined in the domain D. f(z) is said to tend to l as z tends to z_0 in D, if for an $'\in'$ a +ve number howhever small, then there exist δ such that $|f(z)-l| < \in$ as $|z-z_0| < \delta$

i.e $\lim\limits_{z \to z_0} f(z) = l$

Properties of Limits: properties of limit of a complex function f(z) is same as that of the properties of real valued functions.

Problems:

1. Evaluate $\lim\limits_{z \to i} \frac{z^3+i}{5-zi}$

$\lim\limits_{z \to i} \frac{z^3+i}{5-zi} = \frac{i^3+i}{5-i(i)} = \frac{-i+i}{5-i^2} = \frac{0}{6} = 0$

2. Evaluate $\lim\limits_{z \to 2e^{\frac{i\pi}{6}}} \frac{z^2-4}{z^3+z+5}$

consider,

$z = 2e^{\frac{i\pi}{6}} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\frac{1}{2}) = \sqrt{3} + i$

$z^2 = 4e^{\frac{i2\pi}{6}} = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 4(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 2(1 + i\sqrt{3})$

$z^3 = 8e^{\frac{i3\pi}{6}} = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8(0 + i) = 8i$

now, $\lim\limits_{z \to 2e^{\frac{i\pi}{6}}} \frac{z^2-4}{z^3+z+5} = \frac{2(1+i\sqrt{3})-4}{8i+\sqrt{3}+i+5} = \frac{-2(1+i\sqrt{3})}{5+\sqrt{3}+9i}$

3. Evaluate, $\lim\limits_{z \to 2e^{\frac{i\pi}{3}}} \frac{z^3+8}{z^4+4z^2+16}$

consider,

$z = 2e^{\frac{i\pi}{3}} = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 1 + i\sqrt{3}$

$z^2 = 4e^{\frac{i2\pi}{3}} = 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) = 2(\frac{-1}{2} + i\frac{\sqrt{3}}{2}) = -1 + i\sqrt{3}$

$z^3 = 8e^{\frac{i3\pi}{3}} = 8(\cos\pi + i\sin\pi) = 8(-1 + i0) = -8$

$z^4 = 16e^{\frac{i4\pi}{3}} = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = 16(\frac{-1}{2} - i\frac{\sqrt{3}}{2}) = -8(1 + i\sqrt{3})$

now,

$$\lim_{z\to 2e^{\frac{i\pi}{3}}} \frac{z^3+8}{z^4+4z^2+16}=\frac{-8+8}{-8(1+i\sqrt{3})+4(-1+i\sqrt{3})+16}=0$$

4. Evaluate $\lim_{z\to i}\frac{z^2+1}{z^6+1}$

$$\lim_{z\to i}\frac{z^2+1}{z^6+1}=\frac{i^2+1}{i^6+1}=\frac{-1+1}{-1+1}=\frac{0}{0}$$

use L'Hospital's rule

$$\lim_{z\to i}\frac{2z}{6z^5}=\lim_{z\to i}\frac{1}{3z^4}=\frac{1}{3i^4}=\frac{1}{3}$$

5. Evaluate, $\lim_{z\to e^{\frac{i\pi}{3}}}\frac{z\left(z-e^{\frac{i\pi}{3}}\right)}{z^3+1}$

$$\lim_{z\to e^{\frac{i\pi}{3}}}\frac{z\left(z-e^{\frac{i\pi}{3}}\right)}{z^3+1}=\frac{0}{0}$$

$z^3=\left(e^{\frac{i\pi}{3}}\right)^3=\cos\pi+i\sin\pi=-1$

use L'Hospital's rule

$$\lim_{z\to e^{\frac{i\pi}{3}}}\frac{z\left(z-e^{\frac{i\pi}{3}}\right)}{z^3+1}=\lim_{z\to e^{\frac{i\pi}{3}}}\frac{\left(z-e^{\frac{i\pi}{3}}\right)+z}{3z^2}=\frac{0+e^{\frac{i\pi}{3}}}{3e^{\frac{i2\pi}{3}}}=\frac{\frac{1}{2}+i\frac{\sqrt{3}}{2}}{3\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$=\frac{1+i\sqrt{3}}{3(-1+i\sqrt{3})}$$

multiply and divide $-1-i\sqrt{3}$

$$=\frac{1+i\sqrt{3}}{3(-1+i\sqrt{3})}\times\frac{-1-i\sqrt{3}}{-1-i\sqrt{3}}$$

$$=\frac{-1+3-2i\sqrt{3}}{3(1+3)}=\frac{2-2i\sqrt{3}}{12}=\frac{1-i\sqrt{3}}{6}$$

Assignment:

1. Evaluate $\lim_{z\to 1+i}\frac{z^2-z+1-i}{z^2-2z+2}$

Note: $\lim_{z\to z_0} f(z)$ **if exist and it is indipendent of the path as z tends to** z_0**.**

1. prove that $\lim\limits_{z\to i}\frac{\bar{z}}{z}$

$$\lim_{z\to i}\frac{\bar{z}}{z}=\lim_{\substack{x\to 0\\ y\to i}}\frac{x-iy}{x+iy}$$

take the path y=mx

$$=\lim_{\substack{x\to 0\\ y\to i}}\frac{x-imx}{x+imx}=\frac{1-im}{1+im}\text{ ,depends on m}$$

thus, the above limit does not exist.

2. Show that $\lim\limits_{z\to 0}\left(\frac{xy}{x^2+y^2}\right)$ does not exist

$$\lim_{z\to i}\left(\frac{xy}{x^2+y^2}\right)$$

select the path y=mx

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{xy}{x^2+y^2}\right)=\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{x.mx}{x^2+m^2x^2}\right)=\frac{m}{1+m^2}\text{ depends on m}$$

thus,

$\lim\limits_{z\to i}\left(\frac{xy}{x^2+y^2}\right)$ does not exist.

3. Show that $\lim\limits_{z\to 0}\left(\frac{y^2}{x^2+y^2}\right)$ does not exist.

$\lim\limits_{z\to 0}\left(\frac{y^2}{x^2+y^2}\right)$, take tha path along y=x

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\frac{y^2}{x^2+y^2}\right)=\lim_{x\to 0}\left(\frac{x^2}{x^2+x^2}\right)=\lim_{x\to 0}\left(\frac{1}{2}\right)=\frac{1}{2}$$

also, take the path y^2=mx

$$\lim_{z\to 0}\left(\frac{y^2}{x^2+y^2}\right)=\lim_{x\to 0}\left(\frac{mx}{x^2+mx}\right)=\lim_{x\to 0}\left(\frac{m}{x+m}\right)=1$$

the value is not unique, thus the limit does not exist.

Continuity of a complex function:

A complex function f(z) is said to be continuous at $z=z_0$, if $\lim_{z\to z_0} f(z)$ must exit and $\lim_{z\to z_0} f(z)=f(z_0)$

Problems:

1. show that f(z)= $\begin{cases} \dfrac{xy}{x^2+y^2} & \textbf{for } z\neq 0 \\ 0 & \textbf{for } z=0 \end{cases}$ **is not continuous at the origin**

soln: consider, $\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{xy}{x^2+y^2}$

take a path $z\to 0$ along y=mx

$lim_{z\to 0}\frac{xy}{x^2+y^2}=lim_{x\to 0}\frac{xmx}{x^2+m^2x^2}=\frac{m}{1+m^2}$ depending on m

thus, limit does not exist and therefore f(z) is not continuous at z=0

2. show that f(z)= $\begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & \textbf{for } z\neq 0 \\ 1 & \textbf{for } z=0 \end{cases}$ **is not continuous at the origin**

soln: consider, $\lim_{z\to 0} f(z)=lim_{z\to 0}\frac{(x+y)^2}{x^2+y^2}$

consider, alog x-axis, y=0

$lim_{z\to 0}\frac{(x+y)^2}{x^2+y^2}=lim_{x\to 0}\frac{(x+0)^2}{x^2+0}=1$

consider, along y=mx

$lim_{z\to 0}\frac{(x+y)^2}{x^2+y^2}=lim_{x\to 0}\frac{(x+mx)^2}{x^2+m^2x^2}=\frac{1+m}{1+m^2}$ depends on m

thus, limit does not exist and therefore f(z) is not continuous at z=0

Assignment:

1. show that f(z)=$\frac{\bar{z}}{z}$ is discontinuous at the origin.

Differentiation of complex function

Defn: A complex function function f(z) is said to be differentiable at $z=z_o$ if

$$\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$ exists and it is denoted by $f'(z_0)$

i.e $$f'(z_0)=\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$

Above definition can also be defined as

$$f'(z_0)=\lim_{z\to z_0}\left(\frac{f(z_0+\delta z)-f(z_0)}{\delta z}\right)$$

in general, $f(z)$ is differentiable, then

$$f'(z)=\lim_{\delta z\to 0}\left(\frac{f(z+\delta z)-f(z)}{\delta z}\right)$$

Theorem: If f(z) is differentiable at $z=z_0$, then f(z) is continues at $z=z_0$.

Proof:

f(z) is differentiable at $z=z_0$

$$\Longrightarrow f'(z_0)=\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)$$

consider,

$$\lim_{z\to z_0}\left(f(z)-f(z_0)\right)=\lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\times z-z_0\right)$$

$$= f'(z_0).0 = 0$$

$$\lim_{z\to z_0}\left(f(z)-f(z_0)\right)=0$$

$$\lim_{z\to z_0} f(z)=f(z_0)$$

$\Longrightarrow$ f(z) is continuous at $z=z_0$

Analytic Function:

Defn: A complex function f(z) is said to be analytic at $z=z_0$ if it is differentiable not only at $z=z_0$ and also at neibourhood at $z=z_0$.

Analytic function is also called regular function of holomorphic function.

note: sum, product and quotient of to analytic functions is analytic.

Neighbourhood of a point z_0

Neighbourhood of a point $z=z_0$ is the set of points whose centre at $z=z_0$ and radius €, a positive however small.

Necessary and sufficient conditions for f(z) to be analytic

Necessary Condition:

A necessary condition that f(z)=u(x,y)+iv(x,y) be analytic in a domain D is that the 1st order partial derivatives exist and satisfy the equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Proof:

By data f(z) is is analytic in a domain D,

which implies that f(z) is differentiable in D, (analyticity $\Longrightarrow$ differentiability)

i.e $f'(z)=\lim_{\delta z\to 0}\left(\frac{f(z+\delta z)-f(z)}{\delta z}\right)$ exists

the limit exists is unique and it is independent of the path as $\delta z(\delta x,\delta y)\to 0$

i.e $(\delta x,\delta y)\to 0$

we have, f(z)=u(x,y)+iv(x,y)

$$f(z+\delta z)=u(x+\delta x,y+\delta y)+iv(x+\delta x,y+\delta y)$$

$$f(z+\delta z)-f(z)=u(x+\delta x,y+\delta y)+iv(x+\delta x,y+\delta y)-[u(x,y)+iv(x,y)]$$
$$f(z+\delta z)-f(z)=u(x+\delta x,y+\delta y)-u(x,y)+i[v(x+\delta x,y+\delta y)-v(x,y)]$$

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z}$$
$$=\lim_{\substack{\delta x\to 0\\ \delta y\to 0}}\frac{u(x+\delta x,y+\delta y)-u(x,y)+i[v(x+\delta x,y+\delta y)-v(x,y)]}{\delta x+i\delta y}$$

consider along the path, $\delta y=0$ i.e $(\delta x)\to 0$

$$f'(z)=\lim_{\delta x\to 0}\left(\frac{u(x+\delta x,y)-u(x,y)}{\delta x}+i\frac{[v(x+\delta x,y)-v(x,y)]}{\delta x}\right)$$

$$f'(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial x}\right)------(1)$$

Also, consider along the path, $\delta x=0$ i.e $(\delta y)\to 0$

$$f'(z)=\lim_{\delta y\to 0}\left(\frac{u(x,y+\delta y)-u(x,y)}{i\delta y}+i\frac{[v(x,y+\delta y)-v(x,y)]}{i\delta y}\right)$$

$$f'(z)=\lim_{\delta y\to 0}\left(-i.\frac{u(x,y+\delta y)-u(x,y)}{\delta y}+\frac{[v(x,y+\delta y)-v(x,y)]}{\delta y}\right)$$

$$f'(z)=\lim_{\delta y\to 0}\left(-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)----(2)$$

comparing (1) and (2), we get

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{ and }\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$ which are called **Cauchy's-Riemann** equations.

<u>Sufficient condition</u>:

The complex function f(z)=u(x,y)+iv(x,y) with first order derivatives of u and v exist and all are continuous in the domain D satisfying $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=\frac{-\partial u}{\partial y}$ then f(z) is analytic.

We have by Taylor's theorem

u(x+δx,y+δy)=u(x,y)+ $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y$+ higher order derivatives

u(x+δx,y+δy)=u(x,y)+ $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1$

similarly,

v(x+δx,y+δy)=v(x,y)+ $\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y+\varepsilon_2$

f(z+ δz)=u(x+ δx,y+ δy)+i v(x+ δx,y+ δy)

= u(x,y)+ $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1$+i[v(x,y)+ $\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y+\varepsilon_2$]

= u(x,y)+ i v(x,y)+ $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1$+i[v(x,y)+ $\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y$]+ $(\varepsilon_1+i\varepsilon_2)$

f(z+ δz) -f(z)= u(x,y)+ i v(x,y)+ $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1$+i[v(x,y)+ $\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}$ δy]+ $(\varepsilon_1+i\varepsilon_2)$

-[u(x,y)+iv(x,y)]

= $\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\varepsilon_1$+i[v(x,y)+ $\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y$]+ $(\varepsilon_1+i\varepsilon_2)$

$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y]+(\varepsilon_1+i\varepsilon_2)$

$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{-\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y+(\varepsilon_1+i\varepsilon_2)$, using C-R equations

$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+i\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)\delta y+(\varepsilon_1+i\varepsilon_2)$

$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)+(\varepsilon_1+i\varepsilon_2)$

f(z+ δz) -f(z) = $\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z+(\varepsilon_1+i\varepsilon_2)$

divide by δz and take limit as $\delta z\to 0$, then ε_1& ε_2 having higher derivatives are negligible

$$lim_{\delta z\to 0}\left(\frac{f(z+\delta z)-f(z)}{\delta z}\right)=lim_{\delta z\to 0}\frac{\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z}{\delta z}$$

$$f'(z)=lim_{\delta z\to 0}\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)$$

i.e $f'(z)$ exist, therefore f(z) it is analytic in the domain D

Cauchy-Reimann Equations in Polar form

Proof:

Let (r, Θ) be the polar co-ordinates of a point whose cartesian co-ordinates are (x,y), we have x=rcosΘ, y=r sinΘ.

now, z=x+iy= rcosΘ + i rsinΘ = r(cosΘ + i sinΘ) =re$^{i\Theta}$

we have f(z)=f(r e$^{i\Theta}$)

i.e u+iv= f(r e$^{i\Theta}$), where u and v are functions of r and Θ

diff. wr t 'r'

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'(re^{i\theta}).e^{i\theta} \text{------}(1)$$

diff. wr t ' θ '

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'(re^{i\theta}).r.i.e^{i\theta}$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=r.i\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right),\ \text{using (1)}$$

$$\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=ri\frac{\partial u}{\partial r}-r\frac{\partial v}{\partial r}----(2)$$

equating real and imaginary parts

$$\boxed{\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r},\quad \frac{\partial v}{\partial \theta}=r\frac{\partial u}{\partial r} \ \textbf{or}\ \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}}$$ are the C-R

Equations in polar form.

I. Show that every differentiable function in the complex plane is continuous but not converse.

Proof:

Let f(z) is differentiable at z=z_0

i.e $f'(z_0)=lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exist in whathever manner as z approches to z_0

consider, $lim_{z\to z_0}[f(z)-f(z_0)]=lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}.(z-z_0)$

$= f'(z_0)\ \times 0$

=0

$lim_{z\to z_0}[f(z)-f(z_0)]$=0

$lim_{z\to z_0} f(z)= f(z_0)$

thus, f(z) is continous at z_0.

To prove that the converse is not true, i.e every continous function is need be not differentiable

consider, an example **f(z)=$\bar{z}$ is continous at z=0** we have, $f'(0) = lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = lim_{z\to 0}\frac{\bar{z}-0}{z-0}$

$= lim_{z\to 0}\frac{\bar{z}}{z} = lim_{\substack{x\to 0\\ y\to 0}}\frac{x-iy}{x+iy}$

take the path along x-axis(y=0)

$lim_{x\to 0}\frac{x}{x} = 1$

take the path along y-axis(x=0)

$lim_{y\to 0}\frac{-iy}{iy} = -1$

limiting value is different along different paths, thus the limit does not exist.

therefore $f'(0)$ does not exist.

Problems: **Show that the following functions are analytic**

1. $f(z)=z^2$

soln:

$f(z)=(x+iy)^2$

$=x^2+(iy)^2+i2xy$

$=x^2-y^2+i2xy$

$u=x^2-y^2$, $v=2xy$

$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y \quad \frac{\partial v}{\partial x} = 2y, \frac{\partial v}{\partial y} = 2x$

C-R eqns. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$ are satisfied

therefore, $f(z)=z^2$ is analytic

2. $f(z)=e^z$

soln:

$f(z)=e^{x+iy}=e^x e^{iy} = e^x(\cos y + i \sin y)$

$u= e^x\cos y$, $v= e^x\sin y$

$\frac{\partial u}{\partial x} = e^x\cos y, \frac{\partial u}{\partial y} = -e^x\sin y \quad \frac{\partial v}{\partial x} = e^x\sin y, \frac{\partial v}{\partial y} = e^x\cos y$

C-R eqns. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied, therefore f(z) is analytic.

3. f(z)=sinz

Soln: f(z)=sinz

=sin(x+iy)

=sinx cos(iy)+cosx sin(iy)

=sinx coshy + i cosx sinhy, **where cos(iy)=coshy, sin(iy)=isinhy** u=sinxcoshy v=cosx sinhy

$$\frac{\partial u}{\partial x}=\text{cosx coshy}, \frac{\partial u}{\partial y}=\text{sinx sinhy} \quad \frac{\partial v}{\partial x}=\text{-sinx sinhy}, \frac{\partial v}{\partial y}=\text{cosx coshy}$$

C-R eqns. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ are satisfied, therefore f(z) is analytic.

4. f(z)= $\frac{1}{z}$

Soln: f(z)= $\frac{1}{z}$

$$f(z)=\frac{1}{x+iy}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$$

$$=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$

u= $\frac{x}{x^2+y^2}$, v= $\frac{-y}{x^2+y^2}$

$$\frac{\partial u}{\partial x}=\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}, \frac{\partial v}{\partial x}=\frac{y}{(x^2+y^2)^2}.2x=\frac{2xy}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y}=\frac{-x}{(x^2+y^2)^2}.2y=\frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y}=-\left(\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right)=\frac{y^2-x^2}{(x^2+y^2)^2}$$

C-R eqns. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ are satisfied, therefore f(z) is analytic.

Assignment:

5. log(z), 6. cosz, 7. z^3

8. Show that f(z)=z^2+1 is analytic and hence find f'(z)

Soln: f(z)=z^2+1

=$(x+iy)^2$+1

=x^2-y^2+i 2xy +1

u= x^2-y^2+1, v=2xy

$$\frac{\partial u}{\partial x}=2x, \frac{\partial u}{\partial y}=-2y \qquad \frac{\partial v}{\partial x}=2y, \frac{\partial v}{\partial y}=2x$$

C-R eqns. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ are satisfied, therefore f(z) is analytic.

we have,

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+i2y$$

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2(x+iy)=2z$$

9. Show that f(z)=cosz is analytic and hence find f'(z)

Soln: f(z)=cosz

=cos(x+iy)

=cosx cos(iy)-sinx sin(iy)

=cosx coshy - i sinx sinhy

u= cosx coshy, v= -sinx sinhy

$$\frac{\partial u}{\partial x}=\text{-sinx coshy},\ \frac{\partial u}{\partial y}=\text{cosx sinhy}$$

$$\frac{\partial v}{\partial x}=-\text{cosx sinhy},\ \frac{\partial v}{\partial y}=\text{-sinx coshy}$$

C-R eqns. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ are satisfied, therefore f(z) is analytic.

we have

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$f'(z)=\text{-sinx coshy}+i(-\text{cosx sinhy})$$

$$f'(z)=\text{-(sinx coshy}+i\,\text{cosx sinhy)}$$

$$=\text{-(sinx cos(iy) + cosx sin(iy))}$$

$$=\text{- sin(x + iy)}$$

$$=\text{-sinz}$$

10. If f(z)=u+iv is analytic , then show that $\left(\frac{\partial}{\partial x}|f(z)|\right)^2+\left(\frac{\partial}{\partial y}|f(z)|\right)^2=|f'(z)|^2$

Proof:we have, f(z)=u+iv

$$|f(z)|^2=u^2+v^2$$

Diff. w r t x

$$2|f(z)|\frac{\partial}{\partial x}(|f(z)|)=2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}$$

$$|f(z)|\frac{\partial}{\partial x}(|f(z)|)=u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}----(1)$$

similarly,

Diff. w r t y

$$2|f(z)|\frac{\partial}{\partial y}(|f(z)|)=2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}$$

$$|f(z)|\frac{\partial}{\partial y}(|f(z)|)=-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}----(2), \text{using C - R equations}$$

squaring and adding (1) and (2), we have

$$|f(z)|^2\left[\frac{\partial}{\partial x}(|f(z)|)\right]^2+|f(z)|^2\left[\frac{\partial}{\partial y}(|f(z)|)\right]^2=\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)^2+\left(-u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right)^2$$

$$|f(z)|^2\left(\left[\frac{\partial}{\partial x}(|f(z)|)\right]^2+\left[\frac{\partial}{\partial y}(|f(z)|)\right]^2\right)=u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2+2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

$$+u^2\left(\frac{\partial u}{\partial x}\right)^2+v^2\left(\frac{\partial v}{\partial x}\right)^2-2uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$$

$$=(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$|f(z)|^2\left(\left[\frac{\partial}{\partial x}(|f(z)|)\right]^2+\left[\frac{\partial}{\partial y}(|f(z)|)\right]^2\right)=|f(z)|^2|f'(z)|^2, \text{ we have } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

cancelling $|f(z)|^2$ both the sides, we get

$$\left(\frac{\partial}{\partial x}|f(z)|\right)^2+\left(\frac{\partial}{\partial y}|f(z)|\right)^2=|f'(z)|^2$$

11. If f(z)=u+iv is analytic and ϕ is any differential function of x and y, then show that $\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2=\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]|f'(z)|^2$

Proof: By chain rule in differentiation(total derivative), we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}----(1)$$ and

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial u}\left(-\frac{\partial v}{\partial x}\right)+\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x},\ \text{using C - R equations}$$

$$\frac{\partial\phi}{\partial y}=-\frac{\partial\phi}{\partial u}\frac{\partial v}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x}----(2)$$

squaring and adding (1) and (2)

$$\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2=\left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\right)^2+\left(-\frac{\partial\phi}{\partial u}\frac{\partial v}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x}\right)^2$$

$$=\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+2\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}.\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}+\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$-2\frac{\partial\phi}{\partial u}\frac{\partial v}{\partial x}.\frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x}$$

$$=\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial u}\right)^2\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\left(\frac{\partial u}{\partial x}\right)^2$$

$$=\left(\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)$$

$$=\left(\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right)\left|f'(z)\right|^2,\quad \text{where } \left|f'(z)\right|=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Orthogonal system of curves:

Two families of curves $f(x,y)=c_1$ and $g(x,y)=c_2$ are said to be orthogonal families if they intersect right angles to each other.

Thm: If $f(z)= u(x,y) +i\, v(x,y)$ is analytic then $u(x,y)=c_1$ and $v(x,y)=c_2$ are orthogonal families.

Proof: we have

consider, $u(x,y)=c_1$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\frac{dy}{dx}=0\text{, then } m_1=\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial x}}\text{, by C - R Eqn.}$$

also,

v(x,y)=c_2

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0\text{, then } m_2=\frac{dy}{dx}=-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}=\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}}\text{, by C - R Eqn.}$$

now,

$$m_1\times m_2=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial x}}\times\frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}}=-1$$

thus the curves u(x,y)=c_1 and v(x,y)=c_2 are orthogonal family of curves.

Harmonic function:

A function *f*(x,y) is said to be orthogonal, if it satisfies the Laplace's equation

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

Thm: If f(z)= u(x,y) +i v(x,y) is analytic then u(x,y) and v(x,y) are harmonic functions.

Proof: we have, f(z)= u(x,y) +i v(x,y) is analytic

then the C-R equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ - - - (1) and $\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$ - - - (2)are satisfied

Diff.(1) w r t x and (2) w r t y and adding,

we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\quad=0$$

this proves that u(x,y) is harmonic

also,

Diff.(1) w r t y and (2) w r t x and subtracting,

we get

$$\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

this proves that v(x,y) is harmonic

Harmonic conjugates:

Let u(x,y) be harmonic function. If v(x,y) is said to be harmonic conjugate of u then (i) v(x,y) is harmonic and (ii) v satisfies C-R equations.

Thm: If u(x,y) and v(x,y) are harmonic conjugates to each other iff they are constant functions.

Proof: u(x,y) and v(x,y) are harmonic conjugates

then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{---}(1) \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \text{---}(2) \text{ are satisfied}$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{---}(3) \text{ and } \frac{\partial v}{\partial y} = \frac{-\partial u}{\partial x} \text{---}(4) \text{ are satisfied}$$

using (1) and (4), we have

$$\frac{\partial u}{\partial x} = \frac{-\partial u}{\partial x} \Rightarrow 2\frac{\partial u}{\partial x} = 0$$

$\frac{\partial u}{\partial x} = 0$, then u is indipendent of x

using (2) and (3), we have

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} \Rightarrow 2\frac{\partial u}{\partial y} = 0$$

$\frac{\partial u}{\partial y} = 0$, then u is indipendent of y

thus, u is independent of x and y

similarly, we can prove v is independent of x and y conversely, If u=constant, then $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ $\Longrightarrow$ u is harmonic

similarly, If v=constant then $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$ $\Longrightarrow$ u is harmonic

Problems:

1. Prove that $y^3 - 3x^2y$ is harmonic and hence find its conjugate

soln:

Let $u = y^3 - 3x^2y$

$$\frac{\partial u}{\partial x} = -6xy, \quad \frac{\partial u}{\partial y} = 3y^2 - 3x^2$$
$$\frac{\partial^2 u}{\partial x^2} = -6y, \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

then, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which proves u is harmonic

let v be the harmonic conjugate of u

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{, using C - R eqn.}$$
$$dv = -(3y^2 - 3x^2)dx + (-6xy)dy$$
$$dv = (3x^2 - 3y^2)dx + (-6xy)dy$$

this is an exact diff. eqn. of the form M dx + N dy =0 is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Soln. is

$$v = \int M\,dx + \int (\text{terms independent of } x \text{ in N})dy + c$$
$$v = \int (3x^2 - 3y^2)dx + 0.dy + c$$
$$v = x^3 - 3xy^2 + c$$

2. Prove that $\frac{1}{2}\log\left(x^2 + y^2\right)$ is harmonic and hence find its conjugate

Soln: let $u = \frac{1}{2}\log\left(x^2 + y^2\right)$

$$\frac{\partial u}{\partial x}=\frac{1}{2}\frac{1}{\left(x^2+y^2\right)}2x=\frac{x}{x^2+y^2},\quad \frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)1-x.2x}{\left(x^2+y^2\right)^2}=\frac{y^2-x^2}{\left(x^2+y^2\right)^2}--(1)$$

$$\frac{\partial u}{\partial y}=\frac{1}{2}\frac{1}{\left(x^2+y^2\right)}2y==\frac{y}{x^2+y^2}\quad ,\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)1-y.2y}{\left(x^2+y^2\right)^2}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}--(2)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

therefore, u is harmonic.

let v be the harmonic conjugate of u

$$dv=\frac{\partial v}{\partial x}\text{dx}+\frac{\partial v}{\partial y}\text{dy}$$

$$=-\frac{\partial u}{\partial y}\text{dx}+\frac{\partial u}{\partial x}\text{dy}\quad \text{, using C - R eqn.}$$

$$\text{dv}=-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$$

this is an exact diff. eqn. of the form M dx + N dy =0 is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Soln. is

$$\text{v}=\int M\,dx+\int(\text{terms independent of x in N})dy+c$$

$$\text{v}=\int-\frac{y}{x^2+y^2}dx+0.dy+c$$

$$v=-y\tan^{-1}\left(\frac{x}{y}\right)+c$$

3. Prove that e^xcosy + xy is harmonic and hence find its conjugate

soln:

Let u= e^xcosy + xy

$$\frac{\partial u}{\partial x}=\mathbf{e}^x\text{cosy}+\text{y},\quad \frac{\partial u}{\partial y}=-\mathbf{e}^x\text{siny}+\text{x}$$

$$\frac{\partial^2 u}{\partial x^2}=\mathbf{e}^x\text{cosy},\quad \frac{\partial^2 u}{\partial y^2}=-\mathbf{e}^x\text{cosy}$$

then, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

which proves u is harmonic.

let v be the harmonic conjugate of u

$$dv = \frac{\partial v}{\partial x}\mathrm{dx} + \frac{\partial v}{\partial y}\mathrm{dy}$$

$$= -\frac{\partial u}{\partial y}\mathrm{dx} + \frac{\partial u}{\partial x}\mathrm{dy} \quad \text{, using C - R eqn.}$$

$$\mathrm{dv} = (\mathbf{e}^x\mathrm{siny} - \mathrm{x})dx + (\mathbf{e}^x\mathrm{cosy} + \mathrm{y})dy$$

this is an exact diff. eqn. of the form M dx + N dy =0 is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Soln. is

$$\mathrm{v} = \int M\,dx + \int (\text{terms independent of } x \text{ in N})dy + c$$

$$\mathrm{v} = \int (\mathbf{e}^x\mathrm{siny} - \mathrm{x})dx + y.dy + c$$

$$v = \mathbf{e}^x\mathrm{siny} - \frac{x^2}{2} + \frac{y^2}{2} + c$$

Assignment :

Prove that the following functions are harmonic and find its conjugate

(i) x^2-y^2+x+1 (ii) e^xsiny + x^2 - y^2

Construction of analytic functions:

Finding one part(real or imaginary) in which other part of analytic function is given

1. Find the analytic function whose real part is x^3-3xy^2.

soln: let f(z)=u+iv be analytic function.

given u= x^3-3xy^2

To find the imaginary part v

1st method:

we have

$$dv = \frac{\partial v}{\partial x}\mathrm{dx} + \frac{\partial v}{\partial y}\mathrm{dy}$$

$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$, using C-R eqn.

$dv = 6xy\,dx + (3x^2 - 3y^2)dy$ is an exact DE

soln is

$$v = \int M\,dx + \int (\text{terms independent of } x \text{ in N})dy + c$$

$$v = \int 6xy\,dx + \int -3y^2 dy + c$$

$v=3x^2y-y^3+ c$ is the imaginary part.

analytic function is

$f(z)=u+iv$

$= x^3-3xy^2+i(3x^2y-y^3+c)$

$=(x+iy)^3+c$

$=z^3 + c$

Alternate method:(Milne Thomsons' method)

soln: let f(z)=u+iv be analytic function.

given $u= x^3-3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ by C-R eqn.

$$f'(z) = (3x^2 - 3y^2) + i\,6xy$$

By Milne Thomsons' method, put x=z and y=0

we get,

$$f'(z) = 3z^2$$

integrating w r t z

$f(z)=z^3+ c$, is the analytic function.

2. Find the analytic function f(z)=u+iv, whose real part is $u= e^x(x\cos y-y\sin y)$.

Soln:

$u= e^x(x\cos y-y\sin y)$

$$\frac{\partial u}{\partial x} = e^x \cos y + e^x(x\cos y - y\sin y) = e^x(\cos y + x\cos y - y\sin y)$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y) = -e^x(x\sin y + y\cos y + \sin y)$$

we have

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x}$$ by C-R eqn

$$= \frac{\partial u}{\partial x} - \mathrm{i}\,\frac{\partial u}{\partial y}$$

$$f'(z) = e^x(\cos y + x\cos y - y\sin y) - \mathrm{i}\, e^x(x\sin y + y\cos y + \sin y)$$

By Milne Thomsons' method, put x=z and y=0

we get,

$$f'(z) = e^z(1+z)$$

integrating w r t z, we get

$$f(z) = e^z(1+z) - \int e^z \mathrm{d}z$$

$$= e^z(1+z) - e^z + c$$

$$= z\, e^z + c$$

3. Find the analytic function f(z)=u+iv, whose imaginary part is v= xsinxsinhy -ycosxcoshy.

soln:

v= xsinxsinhy -ycosxcoshy

$$\frac{\partial v}{\partial x} = (x\cos x + \sin x)\sinh y + y\sin x\cosh y$$

$$\frac{\partial v}{\partial y} = x\sin x\cosh y - \cos x(y\sinh y + \cosh y)$$

we have

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$$ by C-R eqn

$$= \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x}$$

$$f'(z) = x\sin x\cosh y - \cos x(y\sinh y + \cosh y) + \mathrm{i}\,[(x\cos x + \sin x)\sinh y + y\sin x\cosh y]$$

By Milne Thomsons' method, put x=z and y=0

we get,

$$f'(z) = z\sin z - \cos z$$

integrating w r t z, we get

$$f(z) = -z\cos z + \int \cos z dz - \sin z + c$$

$$= - z\cos z + \sin z - \sin z + c$$

$$= - z\cos z + c$$

4. Find the analytic function f(z)=u+iv, whose imaginary part is $e^{-y}(x\sin x+y\cos x)$

soln:

Given, v= $e^{-y}(x\sin x+y\cos x)$

$$\frac{\partial v}{\partial x} = e^{-y}(x\cos x + \sin x - y\sin x), \quad \frac{\partial v}{\partial y} = e^{-y}\cos x - e^{-y}(x\sin x + y\cos x)$$

$$= e^{-y}(\cos x - x\sin x - y\cos x)$$

we have

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$$ by C-R eqn

$$= \frac{\partial v}{\partial y} + \mathrm{i}\frac{\partial v}{\partial x}$$

$$f'(z)=e^{-y}(\cos x - x\sin x - y\cos x) + i\,e^{-y}(x\cos x + \sin x - y\sin x)$$

By Milne Thomsons' method, put x=z and y=0

we get

$$f'(z)=(\cos z - z\sin z) + i\,(z\cos z + \sin z)$$

integrating w r t z, we get

$$f(z)=\sin z - \left(-z\cos z - \int -\cos z\ dz\right) + i\left(z\sin z - \int \sin z\ dz - \cos z\right)$$

$$= \text{sinz} + \text{z cosz} - \text{sinz} + i(\text{z sinz} + \text{cosz} - \text{cosz})$$

$$= \text{z cosz} + i\,\text{z sinz} \ \mathbf{+c} \ \ \mathbf{=z(cosz+isinz)+c}$$

5. Find the analytic function f(z)=u+iv, given u - v=e^x(cosy-siny)

soln:

u - v=e^x(cosy-siny)

Diff. w r t x

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=e^x(\cos y-\sin y)---(1)$$

Diff. w r t y

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=e^x(-\sin y-\cos y)$$

using C-R equations

$$-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}= -e^x(\sin y+\cos y)----(2)$$

adding (1)and (2), we get

$$-2\frac{\partial v}{\partial x}=-2e^x\sin y$$

$$\frac{\partial v}{\partial x}=e^x\sin y$$

subtracting (1)and (2), we get

$$2\frac{\partial u}{\partial x}=2e^x\cos y$$

$$\frac{\partial u}{\partial x}=e^x\cos y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$$

By Milne Thomsons' method, put x=z and y=0

$$f'(z) = e^z$$

integrating w r t z

$f(z)=e^z+c$

6. Find the analytic function f(z)=u+iv, given u + v= $\frac{x}{x^2+y^2}$

soln:

$$u + v = \frac{x}{x^2+y^2}$$

Diff. w r t x

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} ---(1)$$

Diff. w r t y

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{-x}{(x^2+y^2)^2}(2y)$$

$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} \quad ---(2)$$ Using C-R eqns.

adding (1)and (2), we get

$$2\frac{\partial u}{\partial x} = \frac{y^2-x^2-2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{y^2-x^2-2xy}{\left(x^2+y^2\right)^2}\right]$$

subtracting (1)and (2), we get

$$2\frac{\partial v}{\partial x} = \frac{y^2 - x^2 + 2xy}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{2}\left[\frac{y^2 - x^2 + 2xy}{\left(x^2 + y^2\right)^2}\right]$$

$$f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\,\frac{\partial v}{\partial x}$$

$$= \frac{1}{2}\left[\frac{y^2 - x^2 - 2xy}{\left(x^2 + y^2\right)^2}\right] + i\,\frac{1}{2}\left[\frac{y^2 - x^2 + 2xy}{\left(x^2 + y^2\right)^2}\right]$$

By Milne Thomsons' method, put x=z and y=0

$$f'(z) = \frac{1}{2}\left[\frac{z^2}{z^4}\right] + i\,\frac{1}{2}\left[\frac{-z^2}{z^4}\right] = \frac{1}{2}\left(\frac{1}{z^2} - i\,\frac{1}{z^2}\right) = \left(\frac{1-i}{2}\right)\frac{1}{z^2}$$

on integrating w r t z

$$f(z) = -\left(\frac{1-i}{2}\right)\frac{1}{z} + c$$

Laplace Equation in polar form:

If f is a function of r and Θ, then the equation $\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0$ is called Laplace equation in polar form.

7. Find the analytic function f(z)=u+iv, whose real part is $\left(r + \frac{1}{r}\right)\cos\theta$

soln:

Given u=$\left(r + \frac{1}{r}\right)\cos\theta$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right), \quad \frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta$$

we have,

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\left(\frac{-1}{r}\frac{\partial u}{\partial \theta}\right)\right), \text{ using C - R equation } \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$$

$$= e^{-i\theta}\left(\left(1 - \frac{1}{r^2}\right)\cos\theta + i\frac{1}{r}\left(r + \frac{1}{r}\right)\sin\theta\right)$$

$$= e^{-i\theta}\left(\left(1 - \frac{1}{r^2}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right)$$

by Milne Thomsons' method put $r = z, \theta = 0$

$$f'(z) = 1 - \frac{1}{z^2}$$

integrate w r t z, we get

$$f(z) = z + \frac{1}{z} + c$$

8. Find the analytic function f(z)=u+iv, whose imaginary part is $\frac{-\sin 2\theta}{r^2}$

Given, v= $\frac{-\sin 2\theta}{r^2}$

$$\frac{\partial v}{\partial r} = \frac{2\sin 2\theta}{r^3}, \quad \frac{\partial v}{\partial \theta} = \frac{-2\cos 2\theta}{r^2}$$

we have, $f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$$f'(z) = e^{-i\theta}\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial r}\right), \text{ using C - R equation } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$= e^{-i\theta}\left(\frac{1}{r}\left(\frac{-2\cos 2\theta}{r^2}\right) + i\frac{2\sin 2\theta}{r^3}\right)$$

using Milne Thomsons' method, put $r = z$ and $\theta = 0$

we get,

$$f'(z) = \frac{-2}{z^3}$$

integrating w r t z

$$f(z) = \frac{1}{z^2} + c$$

Complex line integral:

Let f(z) be a continuous function of all points of a smooth curve(contour) C, then $\int_C f(z)dz$ or $\int_a^b f(z)dz$ is called complex line integral of f(z) along C between the points z=a and z=b.

Note: Properties of complex line integrals are similar to that of line integrals of real valued functions.

Problems:

1.Evaluate $\int_C (x^2 - iy^2)dz$ along the parabola y=2x² from (1,2) to (2,8).

soln:

(i) $y=2x^2$

$dy=4x\,dx$

x varies from 1 to 2

$$\int_C (x^2 - iy^2)dz = \int_1^2 (x^2 - iy^2)(dx + idy) = \int_1^2 (x^2 - i(2x^2)^2)(dx + i4xdx)$$

$$= \int_1^2 (x^2 - i4x^4)(1 + i4x)dx = \int_1^2 \left[x^2 + 16x^5 + i(4x^3 - 4x^4)\right]dx$$

$$= \left. \frac{x^3}{3} + 16\frac{x^6}{6} + i\left(4\frac{x^4}{4} - 4\frac{x^5}{5}\right)\right|_1^2 = \frac{8}{3} + \frac{8}{3}.64 - \left(\frac{1}{3} + \frac{8}{3}\right) + i\left[16 - \frac{4}{5}.32 - \left(1 - \frac{4}{5}\right)\right]$$

$$= \frac{8 + 512 - 9}{3} + i\left[\frac{80 - 128 - 1}{5}\right] = \frac{511}{3} - i\frac{49}{5}$$

2.Evaluate $\int_C z^2 dz$ along the straight line from z=0 to z=3+i.

soln:

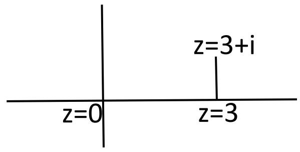
Eqn. of st. line is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \Longrightarrow \frac{x - 0}{3 - 0} = \frac{y - 0}{1 - 0} \Longrightarrow y = \frac{x}{3} \Longrightarrow dy = \frac{dx}{3}$$

$$\int_C z^2 dz = \int_C (x + iy)^2 (dx + idy) = \int_0^3 \left(x + i\frac{x}{3}\right)^2 (dx + \frac{1}{3}dx)$$

$$= \int_0^3 \left(1 + i\frac{1}{3}\right)^2 (1 + i\frac{1}{3})x^2 dx = \left(1 + i\frac{1}{3}\right)^3 \left.\frac{x^3}{3}\right|_0^3 = \frac{(3+i)^3}{27}\frac{27}{3} = \frac{(3+i)^3}{3} = 6 + i\frac{26}{3}$$

3.Evaluate $\int_C z^2 dz$, from z=0 to z=3 and then z=3 to z=3+i.



Soln:

Along z=0=(0,0) to z=3=(3,0)

y=0 and x varies 0 to 3

dy=0

$$\int_C (x+iy)^2(dx+idy) = \int_0^3 x^2 dx = \left.\frac{x^3}{3}\right|_0^3 = 9$$

also, along z=3=(3,0) to z=3+i=(3,1)

x=3 and y varies from 0 to 1

dx=0

$$\int_C z^2 dz = \int_C (x+iy)^2(dx+idy) = \int_0^1 (3+iy)(0+idy) = i\int_0^1 (3+iy)dy = i\left(3y + i\frac{y^2}{2}\right)_0^1$$

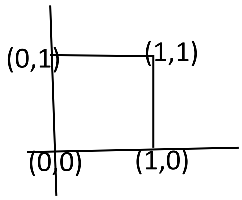
$$= i\left(3 + i\frac{1}{2}\right) = -\frac{1}{2} + 3i$$

$$\int_C z^2 dz = 9 - \frac{1}{2} + 3i = \frac{17}{2} + 3i$$

4.Evaluate $\int_C |z|^2 dz$, where C is the square with vertices (0,0), (1,0), (1,1) and (0,1).

soln:

along the line joining (0,0) to (1,0)



x varies from 0 to 1 and y=0

$$\int_C |z|^2 dz = \int_0^1 \left(x^2 + y^2\right)(dx + idy) = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

along the line joining (1,0) to (1,1)

x =1 and y varies 0 to 1

$$\int_C |z|^2 dz = \int_0^1 \left(x^2 + y^2\right)(dx + idy) = \int_0^1 \left(1 + y^2\right) idy = i\left.\left(y + \frac{y^3}{3}\right)\right|_0^1 = i\left(1 + \frac{1}{3}\right) = \frac{4i}{3}$$

along the line joining (1,1) to (0,1)

y =1 and x varies 1 to 0

$$\int_C |z|^2\, dz = \int_1^0 \left(x^2+y^2\right)(dx+idy) = \int_1^0 \left(x^2+1\right)dx = \frac{x^3}{3}+x\Big|_1^0 = -\left(\frac{1}{3}+1\right) = \frac{-4}{3}$$

along the line joining (0,1) to (0,0)

x =0 and y varies 1 to 0

$$\int_C |z|^2\, dz = \int_1^0 y^2 i dy = i\frac{y^3}{3}\Big|_1^0 = \frac{-i}{3}$$

therefore,

$$\int_C |z|^2\, dz = \frac{1}{3}+\frac{4i}{3}-\frac{4}{3}-\frac{i}{3} = -1+i$$

5.Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx+(2y-x)dy$**, along the parabola** $y^2=x+1$**.**

soln:

$y^2=x+1$ i.e $x=y^2-1$

dx= 2ydy and y varies from 1 to 5

$$\int_{(0,1)}^{(2,5)} (3x+y)dx+(2y-x)dy = \int_1^5 \left(3(y^2-1)+y\right)2ydy+\left(2y-(y^2-1)\right)dy$$

$$=\int_1^5 \left(6y^3+2y^2-6y\right)dy+\left(2y-y^2+1\right)dy = \int_1^5 \left(6y^3+y^2-4y+1\right)dy$$

$$=6\frac{y^4}{4}+\frac{y^3}{3}-4\frac{y^2}{2}+y\Big|_1^5 = \frac{3}{2}5^4+\frac{5^3}{3}-2.5^2+5-\left(\frac{3}{2}+\frac{1}{3}-2+1\right)$$

$$=\frac{1875}{2}+\frac{125}{3}-45-\left(\frac{9+2-6}{6}\right) = \frac{5875}{6}-\frac{5}{6}-45 = \frac{5870}{6}-45 = \frac{2800}{3}$$

6.Evaluate $\int_C (\bar{z})^2 dz$**, around the circle (i)** $|z|=1$ **and (ii)** $|z-1|=1$

soln:

(i) $|z|=1$

or $z=e^{i\theta}$,

$dz=ie^{i\theta}d\theta$, θ varies from 0 to 2π

$\int_C (\bar{z})^2 dz = \int_0^{2\pi} e^{-i2\theta} i\, e^{i\theta} d\theta$, where $\bar{z} = e^{-i\theta}$

$= i\int_0^{2\pi} e^{-i\theta} d\theta = i\left[\frac{e^{-i\theta}}{-i}\right]_0^{2\pi}$

$= -\left[e^{-i2\pi} - 1\right] = 0$, because $e^{-i2\pi} = \cos 2\pi - i\sin 2\pi = 1$

(ii) $|z-1| = 1$

$z-1 = e^{i\theta}$

$z = 1 + e^{i\theta}$, $dz = i\, e^{i\theta} d\theta$ and θ varies from 0 to 2π

$$\int_C (\bar{z})^2 dz = \int_0^{2\pi} \left(1 + e^{-i\theta}\right)^2 i e^{i\theta} d\theta = i\int_0^{2\pi} \left(1 + e^{-2i\theta} + 2e^{-i\theta}\right) e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} \left(e^{i\theta} + e^{-i\theta} + 2\right) d\theta = i\left[\frac{e^{i\theta}}{i} + \frac{e^{-i\theta}}{-i} + 2\theta\right]_0^{2\pi}$$

$$= e^{i2\pi} - e^{-i2\pi} + i4\pi - (1-1) = 1 - 1 + 4\pi i = 4\pi i$$

7.Evaluate $\int_C (x+2y)dx + (4-2x)dy$, around the ellipse $x=4\cos\theta$, $y=3\sin\theta$, $0 \le \theta \le 2\pi$

soln:

$x=4\cos\theta$, $y=3\sin\theta$

$dx=-4\sin\theta\, d\theta$, $dy=3\cos\theta\, d\theta$

$$\int_C (x+2y)dx + (4-2x)dy = \int_0^{2\pi} (4\cos\theta + 6\sin\theta)(-4\sin\theta d\theta) + (4 - 8\cos\theta)3\cos\theta d\theta$$

$$= \int_0^{2\pi} (-16\sin\theta\cos\theta - 24\sin^2\theta + 12\cos\theta - 24\cos^2\theta) d\theta$$

$$= \int_0^{2\pi} (-8\sin 2\theta + 12\cos\theta - 24) d\theta = 4\cos 2\theta + 12\sin\theta - 24\theta\Big|_0^{2\pi}$$

$$= 4\cos 4\pi + 12\sin 2\pi - 48\pi - (4\cos 0 + 12\sin 0 - 0) = 4 - 48\pi - 4 =$$

$$= -48\pi$$

8.Evaluate $\int_C \frac{dz}{z-a}$, around the circle $|z-a| = r$

9.Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$, along the parabola $y=x^2+1$.

Cauchys' Integral Theorem:

statement:

If a function f(z) is analytic at all points within and on a closed contour c, then

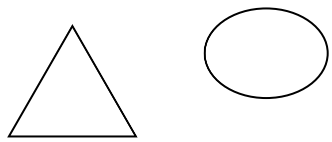
$$\int_C f(z)dz = 0$$

Proof:

Let f(z)=u+iv

let c be the closed contour in the region R

consider,



$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C (udx - vdy) + i(udy + vdx), \textbf{ we have Green's thm: } \int_c Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxd$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy, \text{ using Greens' theorem}$$

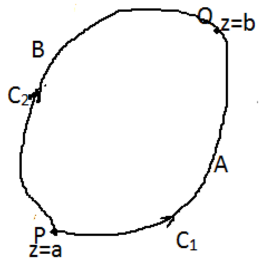
$$= \iint_R \left(-\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dxdy \text{ using C - R eqns.}$$

$$= 0$$

Consequences of Cauchys' Integral Theorem:

I) If f(z) is analytic over a simply connected region R and z=a and z=b are two points in R, then $\int_a^b f(z)dz$ **is always independent of the path joining the points z=a and z=b**

Proof:



Let C consists of two curves C_1 along PQ and C_2 along QP joining z=a at P and z=b at Q in

the region R, then by Cauchys' theorem

$$\int_C f(z)dz = \int_{\text{PAQBP}} f(z)dz = 0$$

$$\int_{\text{PAQ}} f(z)dz + \int_{\text{QBP}} f(z)dz = 0$$

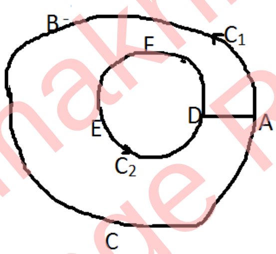
$$\int_{C} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

II) If C_1 and C_2 are two simple closed curves such that C_2 lies completely within C_1. Let f(z) is analytic within and on the boundary of the annular region between C_1 and C_2 then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

Proof: Let C_1 and C_2 be two curves such that C_2 lies completely within C_1. Let us introduce a cut AD connecting A on C_1 and D on C_2.

The curve ABCADEFDA is a simple closed curve and f(z) is analytic inside and on the boundary of C. Hence by Cauchys' integral theorem $\int_C f(z)dz = 0$



Now the region consists of ABCA, AD, DEFD and DA, then

$$\int_{ABCADEFDA} f(z)dz = 0$$

$$\int_{ABCA} f(z)dz + \int_{AD} f(z)dz + \int_{DEFD} f(z)dz + \int_{DA} f(z)dz = 0$$

$$\int_{C_1} f(z)dz + \int_{AD} f(z)dz - \int_{C_2} f(z)dz - \int_{AD} f(z)dz = 0$$

$$\int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0 \implies \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Cauchys' Integral Formula

If f(z) is analytic inside and on a simple closed curve C and 'a' is point within C, then $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

Proof:

Since 'a' is a point within C, consider a circle C_1 with centre at 'a' and radius r>0 and however small.

The function $\frac{f(z)}{z-a}$ is analytic inside and on the annular region between C and C_1

By the II consequence of Cauchys' theorem

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz$$

C_1 is the circle $|z-a| = r$ or $z-a=re^{i\theta}$

$z=a+ re^{i\theta}$

$dz=rie^{i\theta}d\theta$

θ varies around circle from o to 2π

therefore,

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz$$

$$= \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta$$

$$= i\int_0^{2\pi} f(a+re^{i\theta}) d\theta$$

as r tends to zero, then $e^{i\theta}$ also tends to zero

$$\int_C \frac{f(z)}{z-a}dz = i\, f(a) \int_0^{2\pi} d\theta$$

$$= i\, f(a)[\theta]_0^{2\pi}$$

$$= i\, f(a)\, 2\pi = 2\pi i\, f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Generalised of Cauchys' Integral Formula

If f(z) is analytic inside and on a simple closed curve C and 'a' is point within C, then $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

Problems:

1. Evaluate $\int_c \frac{z+4}{z^2+2z+5}dz,$ **where c is** $|z+1-i|=2$**.**

soln: c is the circle $|z-(-1+i)|=2$ centre at (-1,1) and radius=2

consider,

$$\int_c \frac{z+4}{z^2+2z+5}dz = \int_C \frac{z+4}{z^2+2z+1+4}dz$$

$$= \int_C \frac{z+4}{(z+1)^2+4}dz$$

$$= \int_C \frac{z+4}{\left(z+1+2i\right)\left(z+1-2i\right)}dz$$

$$= \int_C \frac{z+4}{\left(z-(-1-2i)\right)\left(z-(-1+2i)\right)}dz$$

clearly, (-1,-2) is an exterior point of C and (-1,2) in interior point of C.

verification: d[(-1,1),(-1,-2)]=3 and d[(-1,1),(-1,2)]=1

$$\int_c \frac{z+4}{z^2+2z+5}dz = \int_C \frac{\frac{z+4}{\left(z-(-1-2i)\right)}}{\left(z-(-1+2i)\right)}dz$$

$=2\pi i\ f(-1+2i)$ using cauchys' integral formula $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

$=2\pi i\ \frac{-i+2i+4}{-1+2i+1+2i}$, where $f(z)=\frac{z+4}{\left(z-(-1-2i)\right)}$

$$=2\pi i\left(\frac{i+4}{4i}\right)=\frac{\pi}{2}(4+i)$$

2. Evaluate $\oint_C \frac{z-1}{(z+1)^2(z-2)}dz,\ where\ c\ is\ |z-i|=2$**.**

soln:

c is the circle centre at (0,1) and radius=2,

$$\oint_C \frac{z-1}{(z-(-1))^2(z-2)}dz$$

clearly, (-1,0) is inside c and (2,0) is outside c

verification: d[(0,1),(-1,0)]=$\sqrt{2}$ <2 and d[(0,1),(2,0)]= $\sqrt{5}$>2

$$\oint_C \frac{z-1}{(z-(-1))^2(z-2)}dz = \oint_C \frac{\left(\frac{z-1}{z-2}\right)}{(z-(-1))^2}dz$$

$$= 2\pi i \ \frac{f'(-1)}{1!}$$, using Cauchys' integral formula

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$= 2\pi i\left(\frac{-1}{(-1-2)^2}\right) = \frac{-2\pi i}{9}, \ where \ \ f(z) = \frac{z-1}{z-2} \ and \ f'(z) = \frac{-1}{(z-2)^2},$$

3. Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz, \ where \ c \ is \ |z| = 3$.

soln:

c is the circle centre at (0,0) and radius=3,

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$$

clearly, both (1,0) and (2,0) is inside 'c'

verification: d[(0,0),(1,0)]=1<3 and d[(0,0),(2,0)]= 2<3

consider,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

$$A = -1$$

$$B = 1$$

$$\oint_C \frac{1}{(z-1)(z-2)}\left(\sin\pi z^2+\cos\pi z^2\right)dz=\oint_C\left(\frac{-1}{(z-1)}+\frac{1}{(z-2)}\right)\left(\sin\pi z^2+\cos\pi z^2\right)dz$$

$$=\oint_C \frac{-(\sin\pi z^2+\cos\pi z^2)}{(z-1)}dz+\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz$$

$$=-2\pi i f(1)\ +2\pi i f(2)$$

$$=-2\pi i(\sin\pi+\cos\pi)+2\pi i(\sin 4\pi+\cos 4\pi)$$

$$=-2\pi i(0-1)+2\pi i(0+1)=4\pi i$$

3. Evaluate $\oint_C \frac{z}{(z^2+1)(z-2)}dz,\ where\ c\ is\ |z|=2$**.**

soln:

C is the circle whose centre at (0,0) and radius=2

$$\oint_C \frac{z}{(z^2+1)(z-2)}dz=\oint_C \frac{z}{(z+i)(z-i)(z-2)}dz$$

$$=\oint_C \frac{z}{(z-(-i))(z-i)(z-2)}dz$$

d[(0,0),(0,-1)]=1<2, d[(0,0),(0,1)]=1<2, d[(0,0),(2,0)]=2=2

i.e, z=-i, z=i are inside C and z=2 is on C

$$\frac{z}{(z-(-i))(z-i)(z-2)}=\frac{A}{(z-(-i))}+\frac{B}{(z-i)}+\frac{C}{(z-2)}$$

$$z=A(z-i)(z-2)+B(z-(-i))(z-2)+C(z-(-i))(z-i)$$

$$put\ z\ =\ -i,\ z\ =i,\ z\ =2,we\ get$$

$$A=\frac{1}{2(i-2)},\ A=\frac{1}{2(i-2)},C=\frac{2}{5}$$

$$\oint_C \frac{z}{(z-(-i))(z-i)(z-2)}\,dz = \oint_C \left(\frac{\frac{1}{2(1-i)}}{(z-(-i))} + \frac{\frac{1}{2(1-i)}}{(z-i)} + \frac{\frac{2}{5}}{(z-2)} \right) dz$$

$$= \frac{1}{2(1-i)} 2\pi i(1) + \frac{1}{2(1-i)} 2\pi i(1) + \frac{2}{5} 2\pi i(1)$$

$$= \frac{2\pi i}{(1-i)} + \frac{4\pi i}{5}$$

Cauchy's Inequality

statement: **If f(z) is analytic inside and on the circle 'C' with centre at z=a and radius 'r' then**

$\left|f^n(a)\right| \leq M\frac{n!}{r^n}$, where n = 0,1,2,3,- - - and M is a positive number such that

$\left|f(z)\right| \leq M$ for all z in 'C'

Proof: We have the generalized Cauchys' integral theorem

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

$$\Rightarrow \left|f^n(a)\right| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz\right|$$

$$= \frac{|n!|}{|2\pi i|}\int_C \frac{|f(z)|}{|z-a|^{n+1}}|dz|$$

$$\leq \frac{n!}{2\pi}M\int_C \frac{1}{r^{n+1}}r.d\theta, \quad z - a = re^{i\theta} \Rightarrow dz = rie^{i\theta}d\theta \Rightarrow |dz| = r\,d\theta$$

$$= \frac{n!}{2\pi r^n}M\int_C d\theta = \frac{n!}{2\pi r^n}M\,\theta\Big|_0^{2\pi} = \frac{Mn!}{r^n}$$

thus, $\left|f^n(a)\right| \leq \frac{Mn!}{r^n}$

Liouvilles' Theorem

statement: **If f(z) is analytic and bounded in the entire complex plane then f(z) is constant.**

Proof:

we have from the Cauchys' inequality

$$\left|f^n(a)\right| \leq \frac{Mn!}{r^n}$$

entire complex plane r → ∞, then $\left|f'(a)\right| \leq 0$, in particular n=1

i.e $f'(a) = 0 \implies f'(z) = 0$

therefore, $f(z) = \text{constant}$

Fundamental Theorem of Algebra

statement: **Every polynomial equation of degree n≥1 with real or complex coefficients has at least one root.**

Proof:

Let $f(z)=a_0+a_1z+a_2z^2+\text{--------}+a_nz^n=0$, ($a_n\neq0$) be a polynomial equation of degree n.

suppose, f(z)=0 has no roots, then f(z) ≠0 for any z

$\emptyset(z)=\frac{1}{f(z)}$ is analytic for all z.

also, $\emptyset(z)=\frac{1}{f(z)}\rightarrow0$ as $z\rightarrow\infty \Longrightarrow$ f(z) is bounded for all z

by Liouvilles' Theorem Ø(z) must be constant, therefore f(z) is a constant function, which contradicts the fact that f(z) is a polynomial of degree n≥1

thus, f(z)=0 for at least a value of z,

f(z) has at least a root.

4. Evaluate $\oint_C \frac{z}{(z^2+1)(z-2)}dz,\ where\ c\ is\ |z|=2$.

soln:

C is the circle whose centre at (0,0) and radius=2

$$\oint_C \frac{z}{(z^2+1)(z-2)}dz = \oint_C \frac{z}{(z+i)(z-i)(z-2)}dz$$

$$= \oint_C \frac{z}{(z-(-i))(z-i)(z-2)}dz$$

d[(0,0),(0,-1)]=1<2, d[(0,0),(0,1)]=1<2, d[(0,0),(2,0)]=2=2

i.e, z=-i, z=i are inside C and z=2 is on C

$$\frac{z}{(z-(-i))(z-i)(z-2)} = \frac{A}{(z-(-i))} + \frac{B}{(z-i)} + \frac{C}{(z-2)}$$

$$z = A(z-i)(z-2) + B(z-(-i))(z-2) + C(z-(-i))(z-i)$$

$$put\ z = -i,\ z = i,\ z = 2, we\ get$$

$$A = \frac{1}{2(i-2)},\ B = \frac{1}{2(i-2)}, C = \frac{2}{5}$$

$$\oint_C \frac{z}{(z-(-i))(z-i)(z-2)}dz = \oint_C \left(\frac{\frac{1}{2(1-i)}}{(z-(-i))} + \frac{\frac{1}{2(1-i)}}{(z-i)} + \frac{\frac{2}{5}}{(z-2)} \right) dz$$

$$= \frac{1}{2(1-i)}2\pi i(1) + \frac{1}{2(1-i)}2\pi i(1) + \frac{2}{5}2\pi i(1),$$ **f(z)=1, in all cases**

$$= \frac{2\pi i}{(1-i)} + \frac{4\pi i}{5}$$

5. Evaluate $\oint_C \frac{z^2-4}{z(z^2+9)}dz,\ where\ c\ is\ |z|=1$.

soln:

C is the circle whose centre at (0,0) and radius=1

consider,

$$\oint_C \frac{z^2-4}{z(z^2+9)}dz = \oint_C \frac{z^2-4}{z(z+3i)(z-3i)}dz$$

now, d[(0,0),(0,0)]=0<1, d[(0,0),(0,-3)]=3>1 and d[(0,0),(0,3)]=3>1

clearly, (0,0) is interior point of C and (0,-3), (0,3) are exterior points of C

$$\oint_C \frac{z^2-4}{z(z^2+9)}dz = \oint_C \frac{z^2-4}{z(z+3i)(z-3i)}dz$$

$$= \oint_C \frac{\frac{z^2-4}{(z+3i)(z-3i)}}{z}dz$$

$=2\pi i\, f(0)$, **where f(z)=** $\frac{z^2-4}{(z+3i)(z-3i)}$, $f(0)=\frac{-4}{9}$

$$=2\pi i \frac{-4}{(3i)(-3i)} = \frac{-8\pi i}{9}$$

6. Evaluate $\oint_C \frac{3z-1}{(z^2-z)}dz$, *where c is* $|z|=2$.

soln:

C is the circle whose centre at (0,0) and radius=2

consider,

$$\oint_C \frac{3z-1}{(z^3-z)}dz = \oint_C \frac{3z-1}{z(z^2-1)}dz = \oint_C \frac{3z-1}{z(z-1)(z+1)}dz$$

z=0,z=1,z=-1 are all interior points of C

d[(0,0),(0,0)]=0<2, d[(0,0),(1,0)]=1<2, d[(0,0),(-1,0)]=1<2

$$\frac{3z-1}{z(z-1)(z+1)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+1}$$

$$3z-1 = A(z-1)(z+1) + Bz(z+1) + Cz(z-1)$$

put $z = 0$, *we get* $A = 1$

Put $z = 1$, *we get* $B = 1$,

put $z = -1$, *we get* $C = -2$

$$\oint_C \frac{3z-1}{z(z-1)(z+1)}dz = \oint_C \left(\frac{1}{z} + \frac{1}{z-1} - \frac{2}{z+1}\right)dz$$

$$= 2\pi i\, f(0) + 2\pi i\, f(1) - 2\pi i\, f(-1) \quad \text{each case } f(z) = 1$$

$$= 2\pi i + 2\pi i - 2\pi i$$

$$=2\pi i$$

7. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^2(z-2)}dz$, *where c is* $|z|=3$.

soln:

C is the circle whose centre at (0,0) and radius=3

consider,

$$\oint_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz$$

d[(0,0),(-1,0)]=1<3, d[(0,0),(2,0)]=2<3

therefore, (-1,0) and (2,0) are interior points of C

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$$

$$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

$$put\ z\ = -1,\ B = \frac{-1}{3}$$

$$put\ z\ = 2,\ C = \frac{1}{9}$$

$$equate\ coefficient\ of\ z^2,\ weget\ 0 = A + C$$

$$A = -C = \frac{-1}{9}$$

$$\oint_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz = \oint_C \left(\frac{\frac{-1}{9}}{(z+1)} + \frac{\frac{-1}{3}}{(z+1)^2} + \frac{\frac{1}{9}}{(z-2)} \right) e^{2z}\,dz$$

$$= \oint_C \left(\frac{-1}{9}\frac{e^{2z}}{(z+1)} - \frac{1}{3}\frac{e^{2z}}{(z+1)^2} + \frac{1}{9}\frac{e^{2z}}{(z-2)} \right) dz$$

$$= \frac{-1}{9} 2\pi i\, f(-1) - \frac{1}{3} 2\pi i\, f'(-1) + \frac{1}{9} 2\pi i\, f(2), \quad where\ f(z) = e^{2z}$$

$$= \frac{-2}{9e^2}\pi i - \frac{4}{3e^2}\pi i + \frac{2}{9} e^4 \pi i$$

Transformations

Let w=f(z) be the complex function. For every point z in the domain there corresponds to unique value f(z) is called the image of z. The domain point/curve in the z-plane gives the corresponding images in the w-plane. The image of every curve in the z-plane in to its image in the w-plane is called transformation.

In the above transformation the mapping of every curve in the z-plane gives the image change its position and magnitude in the w-plane.

The transformation is said to be conformal transformation if the angle between the curves generated at z_0 in the z-plane does not alter the angle in its image in the w-plane.

Some elementary transformations:

1. Reflection:

A transformation w=f(z) is said to be reflection if $f(z)=\bar{z}$, i.e every point (x,y) in z-plane transforms in to (x,-y) in w-plane

2. Translation:

A transformation w=f(z) is said to be reflection if f(z)=z+c , i.e every point (x,y) in z-plane transforms in to $(x+c_1, y+c_2)$ in w-plane

3. Magnification and Rotation:

A transformation w=f(z) is said to be reflection if $f(re^{i\theta})=Re^{i\gamma}$, i.e every point $re^{i\theta}$ in z-plane transforms in to $Re^{i\gamma}$ in w-plane

4. Inversion:

A transformation w=f(z) is said to be reflection if $f(re^{i\theta})=\frac{s}{r}e^{i\theta}$, i.e every point $re^{i\theta}$ in z-plane transforms in to $\frac{s}{r}e^{i\theta}$ in w-plane.

Problems:

1. show that the transformation $f(z)=\frac{1}{z}$ transforms a circle to a circle or to a straight line.

soln:

$$w=\frac{1}{z} \textbf{ or } z=\frac{1}{w}$$

$$x+iy=\frac{1}{u+iv}$$

given $x+iy=\frac{1}{u+iv}\times\frac{u-iv}{u-iv}$

$$=\frac{u-iv}{u^2+v^2}=\frac{u}{u^2+v^2}-i\frac{v}{u^2+v^2}$$

equating real and imaginary parts

$$x=\frac{u}{u^2+v^2}, y=-\frac{v}{u^2+v^2}$$

consider, the standard eqn. of circle in z-plane

$x^2+y^2+2gx+2gy+c=0$

$$\left(\frac{u}{u^2+v^2}\right)^2+\left(-\frac{v}{u^2+v^2}\right)^2+2g\left(\frac{u}{u^2+v^2}\right)+2f\left(-\frac{v}{u^2+v^2}\right)+c=0$$

multiply by $\left(u^2+v^2\right)^2$

$$u^2+v^2+2g\left(u^2+v^2\right)u+2f\left(u^2+v^2\right)v+c\left(u^2+v^2\right)^2=0$$

dividing by $\left(u^2+v^2\right)$

$$1+2gu+2fv+c\left(u^2+v^2\right)=0$$

which represent a circle if c≠0 and straight line if c=0

Special Transformations:

1. Discuss the transformation w=z^2.

soln: $\frac{dw}{dz} = 2z$, the function is analytic for all values of z

consider, w=z^2

u+iv=$(x+iy)^2$

u+iv=x^2-y^2+i2xy

comparing u= x^2-y^2, v=2xy

case(i): when x=k(constant) represents the family of lines parallel to y-axis

u= k^2-y^2---(1), v=2ky----(2)

by eliminating k between (1) and (2)

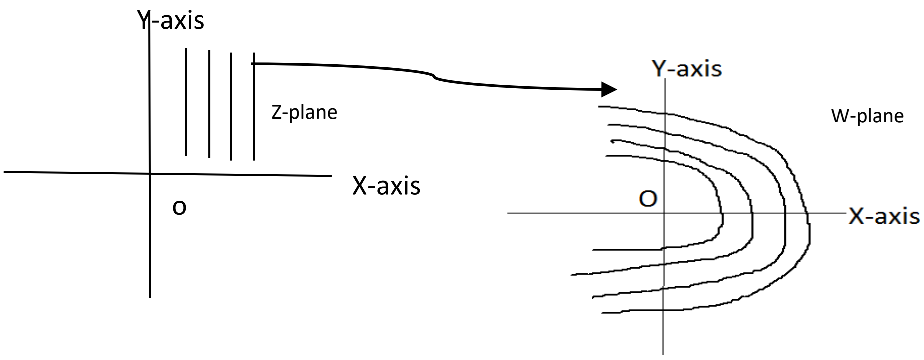
squaring (2) v^2=4$k^2$$y^2$, substituting in (1)

$$u = k^2 - \frac{v^2}{4k^2}$$
$$4k^2u = 4k^4 - v^2$$
$$v^2 = 4k^4 - 4k^2u$$
$$v^2 = -4k^2(u - k^2) - - - -(3)$$

comparing with the parabola $(y-k)^2$=4a(x-h) studied in PUC

eqn.(3), represents parabola symmetric about a line parallel to x-axis.

thus, the lines in the z-plane parallel to x-axis maps on to the parabolas symmetric about the line parallel to y-axis.



case(ii): when y=p(constant) represents the family of lines parallel to X-axis

u= x^2-p^2---(1), v=2xp----(2)

eliminating p between (1) and (2)

$v^2=2x^2p^2$

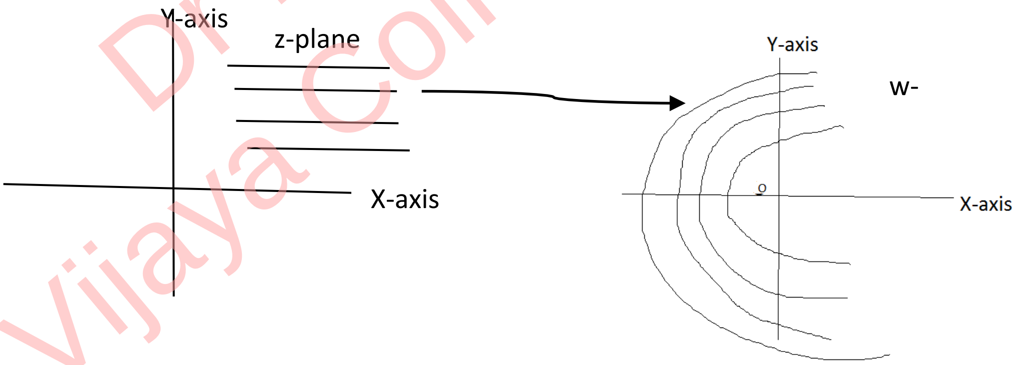
substitute in(1),

$$u=\frac{v^2}{4p^2}-p^2$$
$$4p^2u=v^2-4p^4$$
$$v^2=4p^4+4k^2u$$
$$v^2=4p^2(u+k^2)----(3)$$

comparing with the parabola $(y-k)^2=4a(x-h)$ studied in PUC

eqn.(3), represents parabola symmetric about a line parallel to x-axis.

thus, the lines in the z-plane parallel to x-axis maps on to the parabolas symmetric about the line parallel to y-axis.



1. Discuss the transformation $w=e^z$.

soln: $\frac{dw}{dz}=e^z$, the function is analytic for all values of z

consider, $w=e^z$
$u+iv=e^{x+iy}$
$u+iv=e^xe^{iy}$
$u+iv=e^x(\cos y+i\sin y)$
$u=e^x\cos y$, $y=e^x\sin y$

case(i):

let x=k(constant) the lines parallel to y-axis

$u=e^{k}\cos y$, $y=e^{k}\sin y$

squaring and adding, we get

$u^2+v^2=e^{2k}$

represent the circle in the w-plane

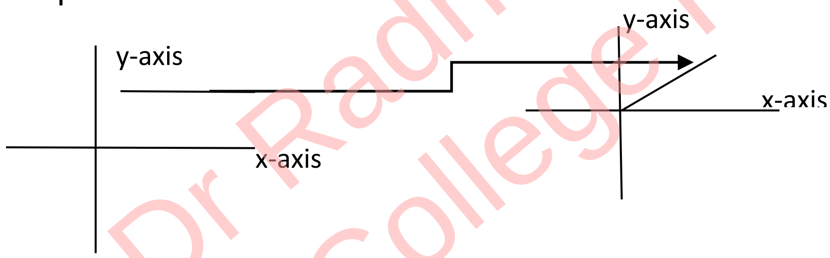
case(ii):

let y=p(constant) the lines parallel to y-axis

$u=e^{x}\cos p$, $v=e^{x}\sin p$

$$\frac{u}{v}=\cot p$$

u=vcotp



In this transformation $w=e^{z}$, the line parallel to y-axis in the z-plane mapping on to circle whose centre at origin in the w-plane and the line parallel to x-axis in the z-plane mapping on to line in the w-plane through origin.

3. Discuss the transformation w=sinz.

soln:

w=sinz

$\frac{dw}{dz} = \cos z,$

$\frac{dw}{dz} = 0$ for z=$\frac{\pi}{2}, \frac{3\pi}{2}$

the function is conformal for all values of z except at $\frac{\pi}{2}, \frac{3\pi}{2}$

put w=u+iv and z=x+iy

u+iv=sin(x+iy)

=sinx cos(iy)+cosx sin(iy)

=sinx coshy + i cosx sinhy

u= sinx coshy, v= cosx sinhy

eliminating 'x'

$\frac{u}{\cosh y} = \sin x, \ \frac{v}{\sinh y} = \cos x$

squaring and adding

$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$

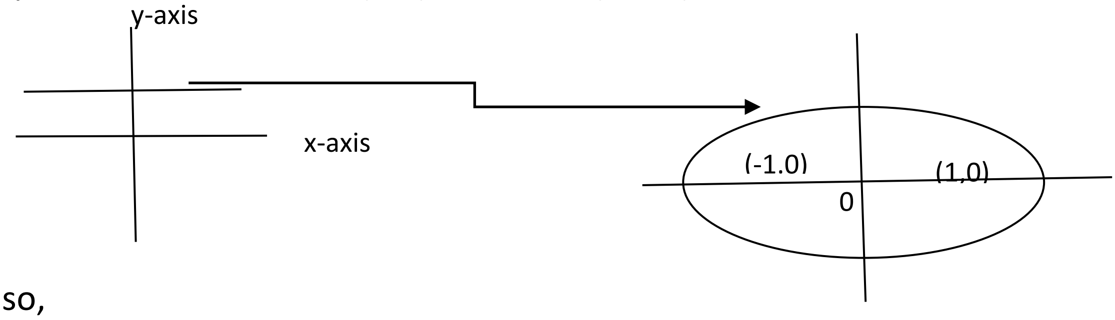
for y=k(constant) represent the line parallel to x-axis

$\frac{u^2}{\cosh^2 k} + \frac{v^2}{\sinh^2 k} = 1$,, represents ellipse whose centre=(0,0)

eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{\cosh^2 k - \sinh^2 k}{\cosh^2 k}} = \frac{1}{\cosh k} = \sec hk$

foci=($\pm$ae,0)=($\pm$1,0)

thus, the family of lines parallel to x-axis in the z-plane mapping on to ellipse in the w-plane whose centre at (0,0) and focus ($\pm$1,0)



also,

eliminating 'y'

$$\frac{u}{\sin x} = \cosh y, \ \frac{v}{\cos x} = \sinh y$$

squaring and subtracting

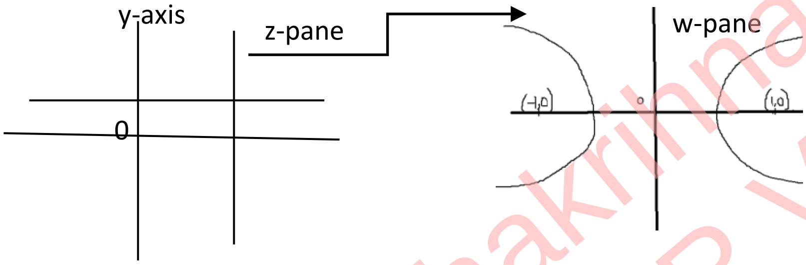
$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

let x= λ, is the line parallel to y-axis.

$\frac{u^2}{\sin^2 \lambda} - \frac{v^2}{\cos^2 \lambda} = 1$, represents hyperbola whose centre=(0,0)

eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{\sin^2 \lambda + \cos^2 \lambda}{\sin^2 \lambda}} = \frac{1}{\sin \lambda} = \operatorname{cosec} \lambda$

foci=($\pm$ae,0)=(± 1,0)



4. Discuss the transformation w=cosz.

soln:

w=cosz

$$\frac{dw}{dz} = -\sin z,$$

$\frac{dw}{dz} = 0$ for z=0, π

the function is conformal for all values of z except at z=0, π

put w=u+iv and z=x+iy

u+iv=cos(x+iy)

=cosx cos(iy)+sinx sin(iy)

=cosx coshy + i sinx sinhy

u= cosx coshy, v= sinx sinhy

eliminating 'x'

$$\frac{u}{\cosh y} = \cos x, \ \frac{v}{\sinh y} = \sin x$$

squaring and adding

$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$, represents ellipse whose centre=(0,0) eccentricity, $e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{\cosh^2 y-\sinh^2 y}{\cosh^2 y}}=\frac{1}{\cosh y}=\sec hy$

foci=($\pm$ae,0)=($\pm$1,0)

also,

eliminating 'y'

$\frac{u}{\sin x}=\cosh y,\ \frac{v}{\cos x}=\sinh y$

squaring and subtracting

$\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}=1$, represents hyperbola whose centre=(0,0)

eccentricity, $e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}=\frac{1}{\sin x}=\cos ecx$

foci=($\pm$ae,0)=($\pm$1,0)

thus, the family of lines parallel to y-axis in the z-plane mapping on to hyperbola in the w-plane whose centre at (0,0) and focus ($\pm$1,0).

5. Discuss the transformation w=$\frac{1}{2}\left(z+\frac{1}{z}\right)$.

soln:

w=$\frac{1}{2}\left(z+\frac{1}{z}\right)$

$\frac{dw}{dz}=\frac{1}{2}\left(1-\frac{1}{z^2}\right)$

$\frac{dw}{dz}=0$ for $z=\pm 1$

therefore the function is conformal at all points except at $z=\pm 1$

Let w=u+iv and z=re$^{i\theta}$

then,

u+iv=$\frac{1}{2}\left(re^{i\theta}+\frac{1}{r}e^{-i\theta}\right)$

=$\frac{1}{2}\left(r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)\right)$

=$\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta+i\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta$

comparing, we get

u= $\frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta$ v= $\frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta$ -------(1)

Cas(i):

Eliminating Θ in (1) and (2), we have

$$\frac{u}{\frac{1}{2}\left(r+\frac{1}{r}\right)}=\cos\theta,\quad \frac{u}{\frac{1}{2}\left(r-\frac{1}{r}\right)}=\sin\theta$$

squaring and adding, we get

$$\frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2}+\frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2}=1,$$

this equation represents ellipse whose centre=(0,0) when r>1.

the circle $|z|$=r in the z-plane maps on to ellipse in the w-plane.

Case (ii):

also, eliminate 'r' between (1) and (2)

$$\frac{u}{\cos\theta}=\frac{1}{2}\left(r+\frac{1}{r}\right)\quad \frac{v}{\sin\theta}=\frac{1}{2}\left(r-\frac{1}{r}\right)$$

$$\frac{u^2}{\cos^2\theta}-\frac{v^2}{sin^2\theta}=\frac{1}{4}\left(r+\frac{1}{r}\right)^2-\frac{1}{4}\left(r-\frac{1}{r}\right)^2$$

$$=\frac{1}{4}\left(\left(r+\frac{1}{r}\right)^2-\left(r-\frac{1}{r}\right)^2\right)$$

$$\frac{u^2}{\cos^2\theta}-\frac{v^2}{sin^2\theta}=1$$

which represents hyperbola whose centre=(0,0) and eccentricity

$$e=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}}=\sqrt{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}=\frac{1}{\cos\theta}$$

foci= $(\pm ae,0)=(\pm 1,0)$

i.e Every constant angle Θ=k in the z-plane is mapping on to the hyperbola in the w-plane whose centre=(0,0) and foci=($\pm$1,0).