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## TOPIC: NORMAL SUBGROUPS

**Defn: A subgroup H of a group is said to normal subgroup of G iff $ghg^{-1}$€H, ∀g€G and h€H.**

**Note: If H is a normal subgroup of a group G, then it is denoted by HΔG**

EX-1: we have H={-1,1} is a subgroup of a group G={-1,1,i,-i} under regular multiplication,

Then, -1(-1)1=1€H, 1(-1)(-1)=1€H, -1(1)1=-1€H, 1(1)(-1)=-1€H

-i(-1)i=-1€H, i(-1)(-i)=-1€H, -i(1)(i)=1€H, i(1)(-i)=1€H

Which proves HΔG

**Alternate definitions of normal subgroup:**

**I.A subgroup H of a group is a normal subgroup of G iff $gHg^{-1}$ =H**

Proof:

Let H be a subgroup of G

⇒ $ghg^{-1}$€H,

let x € $gHg^{-1}$

such that x= $ghg^{-1}$ for some h€H

⇒ x € H

thus, $gHg^{-1}$ C H ------(1)

further, replace g by $g^{-1}$

$g^{-1}H(g^{-1})^{-1}$ C H

i.e $g^{-1}Hg$ C H

pre-operate g and post-operate $g^{-1}$ both the sides

$(g)g^{-1}Hg(g^{-1})$ C $gHg^{-1}$

H C $gHg^{-1}$ ------(2)

from (1) and (2)

$gHg^{-1}$ =H

**II. A subgroup H of a group is a normal subgroup of G iff every right coset is a left coset.**

Proof:

Let H be a subgroup of G

⇒ $gHg^{-1}$ =H,

post operate g

$gHg^{-1}g=Hg$

$gH=Hg$

i.e every left coset is right coset

conversely,

suppose, every left coset is right coset of H in G

Hg=xH

clearly, g € Hg ⟹ g € xH (Hg=xH)

now, g € xH and g € gH

gH= xH (since any to left cosets are either identical or disjoint)

⟹ gH=Hg

$gHg^{-1}=H$

**H∆G**

**III. A subgroup H of a group is a normal subgroup of G iff product of any two right(left) coset is a right(left) coset.**

Proof:

Let H be a subgroup of G

⟹ Let Ha and Hb be to right cosets of H in G

consider, Ha.Hb=H(aH)b

=H(Ha)b

=HH(ab) is a right coset

conversely,

product of any two right coset is a right coset

let Hx and $Hx^{-1}$ be to right cosets of H in G

$Hx.Hx^{-1}$ is again a right coset

$Hx.Hx^{-1}=H$ , since H is both right coset and left coset

let $h_1xh_2x^{-1}$€H

$xh_2x^{-1}$€$(h_1)^{-1}$ H

$xh_2x^{-1}$€ H

**H∆G**

<u>**Theorems on Normal Subgroups**</u>:

**Thm1: The product of any two normal subgroups is a normal subgroup**

Proof: Let H and K be two normal subgroups of G

Define, HK={hk/h€H, k€K}

Let g€G, then $ghg^{-1}$€H where h€H, since H is normal in G

$gkg^{-1}$€K, where k€K, since K is normal in G

now, $ghg^{-1}gkg^{-1}=gh(g^{-1}g)kg^{-1}=ghkg^{-1}$€HK

by defn. of normal subgroup

HK is normal subgroup in G.

**Thm2: The intersection of two normal subgroups is a normal subgroup**

Ans: Let H and K be two normal subgroups of a group G

Clearly, H∩K≠ ∅ as they are subgroups of G

clearly, H∩K is a subgroup of G(proved in 2nd semester)

Let g€ G and h€H∩K

$ghg^{-1}$€H, where h€H

and $ghg^{-1}$€K, where h€K

$\Longrightarrow$ $ghg^{-1}$€H∩K

which proves that H∩K is normal in G.

**Thm3: Let H be a subgroup of G and K is normal subgroup of G, then H∩K is a normal subgroup of H**

Proof: Clearly, H∩K≠ ∅ as they are subgroups of G

clearly, H∩K is a subgroup of G(**proved in 2nd semester**)

Let g€H and h€H∩K

$ghg^{-1}$€K as K is normal in G

gh€H, by closure axiom

$ghg^{-1}$€H, as $g^{-1}$€H

$ghg^{-1}$€H∩K

which proves that H∩K is normal in H.

**Thm4: Let H be a normal subgroup of G and K be any subgroup of G, then HK is a subgroup of G**

Proof:

Define, HK={hk/h€H, k€K}

Clearly, HK$\neq \emptyset$

Let x,y€HK, where $x=h_1k_1$ and $y=h_2k_2$

Now, $xy^{-1}=h_1k_1(h_2k_2)^{-1}$

$= h_1k_1k_2^{-1}h_2^{-1}$

$= h_1(k_1k_2^{-1})h_2^{-1}$

$= h_1k_3\, h_2^{-1}$, where $k_1k_2^{-1}=k_3$€K

$= h_1k_3\, h_2^{-1}(k_3^{-1}k_3)$

$= h_1(k_3\, h_2^{-1}k_3^{-1})k_3$

$= h_1h_3k_3$, where $k_3\, h_2^{-1}k_3^{-1}=h_3$€H as H is normal in G

$= h_4k_3$€HK, where $h_1h_3=h_4$€H

Thus, HK is a subgroup of G.

**CENTRE OF A GROUP**

Defn: Let G be a group. the set of Z of elements of G commutes with every elements of G is called centre of G.

i.e Z={z€G/xz=zx ∀x€G }

**Thm: Centre of a group G is a normal sub-group of G**

Proof: Let Z={z€G/xz=zx ∀x€G } is the centre of G

ley $z_1,z_2$ €Z, so that $xz_1=z_1x$ and $xz_2=z_2x$

consider, $xz_2=z_2x$

$(xz_2)z_2^{-1}=z_2xz_2^{-1}$

$x(z_2z_2^{-1})=z_2xz_2^{-1}$

$xe= z_2xz_2^{-1}$

$x= z_2xz_2^{-1}$

$z_2^{-1}x= xz_2^{-1}$--------(1) by pre operating $z_2^{-1}$

$z_1z_2^{-1}x= z_1xz_2^{-1}$ using (1)

$z_1z_2^{-1}x= xz_1z_2^{-1}$, since $xz_1=z_1x$

$\Longrightarrow$ $z_1z_2^{-1}$€Z

therefore Z is a sub-group of G

also, $xzx^{-1}=zxx^{-1}$ as xz=zx

=ze =z€Z

thus Z is normal subgroup of G

**NORMALIZER OF A GROUP**

Defn: Let G be a group and 'a' be an element of G. The set of elements of G commutes 'a' is called normalizer or centralizer of an element 'a' and it is denoted by Na.

i.e Na={x€G/xa=ax ∀a€G }

**QUOTIENT GROUP**

Let G be a group and H be the normal subgroup of G. The set of all cosets(both left cosets and right cosets) of H in G is called quotient set. This set is denoted by G|H

i.e G|H={Ha/a€G}

**Thm: The set G|H of all cosets of H in G is a group under the composition HaHb=Hab∀a,b€G**

Proof:

(i) Closure axiom:

let Ha,Hb€G|H, then HaHb=Hab€G|H

(ii) Associative axiom:

let Ha,Hb,Hc €G|H, then Ha(HbHc)=HaHbc=Habc

also, (HaHb)Hc=HabHc=Habc

(iii) Identity axiom:

He=H is the identity element in G|H

(iv) Inverse axiom:

let Ha€G|H be any element, then $Ha^{-1}$€G|H such that

$HaHa^{-1}=Haa^{-1}=He=H$, then $Ha^{-1}$ is inverse of Ha.

therefore, G|H is a group

**Defn:** Let G be a group and H be the <u>normal subgroup</u> of G, the quotient set G|H of all cosets is a group under the composition HaHb=Hab is called quotient group.

Problems:

**1. Let h and K be two normal subgroups of G such that H∩K={e} then show that every element of H commutes with every element of K.**

**Soln**: let h€H and k€K be arbitrary

then we shall prove that hk=kh

H is normal in G $\Longrightarrow$ $kh^{-1}k^{-1}$€H

$hkh^{-1}k^{-1}$€H, by closure axiom in H

K is normal in G $\Longrightarrow$ $hkh^{-1}$€K

$hkh^{-1}k^{-1}$€K, by closure axiom in K

thus, $hkh^{-1}k^{-1}$€ H∩K

but H∩K={e}

therefore, $hkh^{-1}k^{-1}=e$

$hkh^{-1}=k$, by post operating k

hk=kh, by post operating h

**2.Prove that every quotient group of an abelian group is abelian**

**soln:**

Let G|H be a quotient group of an abelian group G

Ha,Hb are any two elements of G|H

HaHb=Hab=Hba, as G is abelian

=HbHa

therefore, G|H is an abelian group.

**HOMOMORPHISM OF GROUPS**

**Defn**: Let $(G,*)$ and $(G', o)$ be two groups, the mapping f: $(G,*)\rightarrow (G', o)$ is said to be homomorphism of the group G in to $G'$ if $f(a*b)=f(a)of(b)$, $\forall a,b\in G$.

Ex:

**1. Let (Z,+) and ($Q_0$,·) be the groups where Z is set of integers and $Q_0$ is the set of rationals excluding 0. The mapping f: (Z,+) →($Q_0$,·) defined by $f(a)=2^a$, ∀a€Z, verify f is a homomorphism.**

soln:

let a,b€Z,

$f(a+b)=2^{a+b}=2^a.2^b=f(a).f(b)$

therefore f is a homomorphism.

**2. Show that the mapping f:(R,+) →($R^+$,·) defined by $f(x)=e^x$, ∀x€R, where $R^+$ is the set of positive numbers. Show that f is a homomorphism.**

soln:

let x,y€R,

$f(x+y)=e^{x+y}=e^x.e^y=f(x).f(y)$

therefore f is a homomorphism.

**3. Let (G,+) be a group w r t addition. The mapping f: (G,+) →(G,+) defined by f(x)=x+3, ∀x€G.**

Soln:

let x,y€G,

$f(x+y)=x+y+3$

$\quad =x+3+y$

$\quad \neq f(x)+f(y)$

therefore f is not a homomorphism.

**Thm:1. Let f: G→ $G'$be homomorphism from the group G in to $G'$, then**

**(i) f(e)= $e'$, where e and $e'$ are the identities in G and $G'$**

**(ii) $f(a^{-1})=[f(a)]^{-1}$, ∀a€G**

Proof:

(i) f(a) $e'$=f(a), by identity axiom in $G'$

f(a) $e'$=f(ae)=f(a)f(e), f is a homomorphism

$e'$= f(e)

(ii) consider, $f(a)f(a^{-1})=f(aa^{-1})=f(e)= e'$-----(1)

also, $f(a^{-1})f(a) =f(a^{-1}a)=f(e)= e'$ ------(2)

from (1) and (2) f(a) and $f(a)^{-1}$ are inverses to each other therefore, $[f(a)]^{-1}=f(a^{-1})$.

**Thm:2. Let f:G→ G′be homomorphism from the group G in to G′, then the set f(G)={f(g)/∀g€G} is a subgroup of G′.**

Proof:

let x,y€f(G) such that $x=f(g_1)$, $y=f(g_2)$, where $g_1$and $g_2$ are in G

consider, $xy^{-1}= f(g_1)[f(g_2)]^{-1}$

$= f(g_1)\, f(g_2^{-1})$, using Thm:1

$= f(g_1g_2^{-1})$, since f is a homomorphism

$= f(g_1g_2^{-1})$€f(G) as $g_1g_2^{-1}$€G

therefore, f(G) is a subgroup of G′

**Note**: f(G) is called the homomorphic image of G

**Thm:3. Let f: G→ G′be homomorphism from the group G in to G′ and H is a subgroup of G then f(H) is a subgroup of G′.**

Proof:

Define f(H)={f(h)/∀h€H} is clearly nonempty set

let x,y€f(H) such that $x=f(h_1)$, $y=f(h_2)$, where $h_1$and $h_2$ are in H

consider, $xy^{-1}= f(h_1)[f(h_2)]^{-1}$

$= f(h_1)\, f(h_2^{-1})$, using Thm:1

$= f(h_1h_2^{-1})$, since f is a homomorphism

$= f(h_1h_2^{-1})$€f(H) as $h_1h_2^{-1}$€H

therefore, f(H) is a subgroup of G′

**Thm:4.The homomorphic image of an abelian group is abelian.**

Proof:

Let f: G→ G′be homomorphism from the group G in to G′ and G is abelian.

we have, f(G) is homomorpmic image of G

let x,y€f(G) such that $x=f(g_1)$, $y=f(g_2)$, where $g_1$and $g_2$ are in G

consider, $xy= f(g_1)f(g_2)$

$=f(g_1g_2)$, as f is a homomorphism

$=f(g_2g_1)$, G is abelian

$= f(g_2)f(g_1)$

= yx

therefore f(G) is abelien.

**Thm:5. Let f: G→ G be homomorphism from the group G in to itself and H is a cyclic subgroup of G, the f(H) is also cyclic subgroup of G.**

Proof:

let H=<a>

we shall prove that f(H) is also cyclic generated by f(a)

let $x=a^{m}$€H, y=f(x)€f(H)

case(i): m is any +ve integer

$y=f(a^{m})$

=f(a.a.a.a.....m times)

=f(a)f(a)f(a).......m times, because f is a homomorphism

$=[f(a)]^{m}$,

case(ii): m is any -ve integer

let m=-k

$y=f(a^{-k})$

$= f(a^{-1})^{k}$

$= [f(a^{-1})]^{k}$

$= f(a^{-1}).\ f(a^{-1}).\ f(a^{-1}).\ f(a^{-1})$....k times

$=f(a^{-1}\ a^{-1}\ a^{-1}\ a^{-1}$.....k times)

$= f(a^{-1})^{k}$

$= [f(a)]^{-k}$

thus, f(H)=<f(a)>

i.e f(H) is cyclic generated by f(a)