Syllabus : OSCILLATIONS : *SHM ; Differential equation of SHM and its solutions, Kinetic and Potential energy, Simple and compound pendulum; oscillations of two masses connected by a spring; damped oscillations – over damped, under damped and un-damped oscillations; forced oscillations - concept of resonance; Coupled Oscillators - in phase and out of phase oscillations- energy transfer*

Periodic Motion : A motion that repeats itself in equal intervals of time is called periodic motion. It is also called harmonic motion.

Examples. 1. Revolution of earth round the sun or about its own axis, 2. Revolution of moon and other artificial satellites around the earth, 3. Rotation of the electrons round the nucleus of an atom, 4. The motion of pendulum of a clock, 5. Motion of prongs of a tuning fork and 6. Oscillations of a loaded spring.

The two common periodic motions are the **simple harmonic motion** and **uniform circular motion**.

Oscillatory Motion : The motion of a particle is oscillatory if it moves back and forth or to and fro about the same path at equal intervals of time.

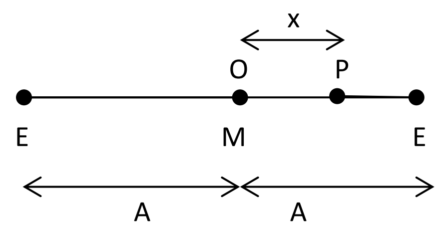
Examples. 1. Motion of a pendulum, 2. Motion of mass attached to a suspended spring, 3. Motion of atoms in a molecule or a lattice, 4. Motion of particles of the medium through which sound travels, 5. Up and down motion of a floating object on water when waves propagate through it, 6. Motion of prongs of a tuning fork.

All oscillatory motions are periodic in nature but all periodic motions are not oscillatory. For example, revolution of earth round the sun is periodic but not oscillatory and Rotation of the electrons round the nucleus of an atom is periodic but not oscillatory.

Definitions:

1. *The smallest time interval in which the motion repeats is called the **period (T).** It is the time taken for one oscillation.*
2. *The number of repetitions of motion that occur per second is called the **frequency (f)** of the periodic motion. Frequency is equal to the reciprocal of period.* $f = \frac{1}{T}$.
3. *The maximum displacement of a particle from its mean position during an oscillation is called **amplitude (A)** of oscillation.*
4. ***Phase :*** *It represents the state of motion of a particle in periodic motion. It is expressed in terms of fraction of period T or the fraction of angle 2π measured from the instant when the body has crossed the mean position in the positive direction.*

Simple Harmonic Motion (SHM)



Consider a particle executing oscillatory motion about the mean position (E – M – E) with E as the extreme position and M is the mean position) with amplitude A. Let x be the displacement of the particle at an instant of time t. A restoring force F acts on the particle to bring it back to its mean position. This force is directly proportional to the displacement.

Mathematically $\boldsymbol{F} \propto \boldsymbol{x}$ or $\boldsymbol{F} = -\boldsymbol{kx}$ ….(1) where k is a constant called the force constant. The negative sign indicates that F acts opposite to the direction of motion of the particle.

As $F = ma$ …..(2) From (1) and (2) $ma = -kx$ *or* $a = -\left(\frac{k}{m}\right)x$ *or* $\boldsymbol{a} \propto \boldsymbol{x}$ **.**

A particle is said to execute simple harmonic motion if the acceleration of the particle is directly proportional to the displacement of the particle from the mean position and it is directed towards the mean or equilibrium position.

Examples of SHM : Motion of a pendulum, motion of mass attached to a suspended spring, Vibrations of a guitar string, etc…

Differential form of SHM : Consider a particle executing SHM. If x is the displacement of the particle from the mean position at an instant of time t,

Then, the restoring force is $F \propto x$ or $F = -kx$ ….(1)

As $F = ma$, ….(2) Comparing (1) and (2) m$a = -kx$ *or* $a = -\left(\frac{k}{m}\right)x$ ….(3)

As $a = \frac{d^2x}{dt^2}$, equation (3) is, $\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$ ……(4) or $\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$ …..(5)

let $\frac{k}{m} = \omega^2$ where ω is called the angular velocity or angular frequency of oscillating particle.

Now, equation (5) is $\boldsymbol{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ ………(6) Equation (6) is called differential form of SHM.

Solution of the differential equation of SHM

1. Expression for displacement of the particle

The differential form of SHM is $\boldsymbol{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ …….(1)

or $\boldsymbol{\frac{d^2x}{dt^2} = -\omega^2 x}$ Multiplying both the sides of this equation by $2\frac{dx}{dt}$, we get

$2\frac{dx}{dt}\boldsymbol{\frac{d^2x}{dt^2}} = -2\frac{dx}{dt}\boldsymbol{\omega^2 x}$. This equation can be expressed as $\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = -\omega^2 \frac{d}{dt}(x)^2$ ...(2)

Integrating equation (2) we get $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + C$ .....(3) where C is the constant of integration.

When the displacement is maximum, i.e. $x = A$, the velocity of the particle $\frac{dx}{dt} = v = 0$

Putting this condition in (3), $0 = -\omega^2 A^2 + C$ or $C = \omega^2 A^2$

Now the equation (3) becomes $\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + \omega^2 A^2$

or $\left(\frac{dx}{dt}\right)^2 = \omega^2(A^2 - x^2)$ or $\boldsymbol{\frac{dx}{dt} = \omega\sqrt{(A^2 - x^2)}}$ ....(4) *(This is also the expression for the velocity of the particle executing SHM)*

Rewriting equation (4) as, $\frac{dx}{\sqrt{(A^2 - x^2)}} = \omega\, dt$.

Integrating this equation, $\int \frac{dx}{\sqrt{(A^2 - x^2)}} = \omega \int dt$ gives $sin^{-1}\left(\frac{x}{A}\right) = \omega t + \varphi$

where φ is the constant called the initial phase of the particle executing SHM. It is also called epoch.

or $\left(\frac{x}{A}\right) = sin\,(\omega t + \varphi)$ or $\boldsymbol{x = A\,sin\,(\omega t + \varphi)}$ **....(5)**

This is the solution of differential equation of SHM (eqn. (1)). Equation (5) is the expression for the displacement of the particle at time t.

2. Expression for Velocity of the particle executing SHM

The equation of SHM is given by $x = A\,sin\,(\omega t + \varphi)$

If the initial position from where time is measured is the mean position, then $\varphi = 0$

Thus $x = A\,sin\omega t$. The velocity of the particle at a given instant of time t is given by differentiating the above equation, i.e. $\frac{dx}{dt} = A\omega\,cos\omega t$ or $\frac{dx}{dt} = A\omega\sqrt{1 - sin^2\omega t}$

As $sin\omega t = \frac{x}{A}$, we have $\frac{dx}{dt} = A\omega\sqrt{1 - \frac{x^2}{A^2}}$ or $\frac{dx}{dt} = A\omega\sqrt{\frac{A^2 - x^2}{A^2}}$

Thus the velocity of the particle is $\boldsymbol{v = \frac{dx}{dt} = \omega\sqrt{A^2 - x^2}}$

Note : *At the extreme position of oscillation, when* $x = A$ *,* $\boldsymbol{v = 0}$*. At the mean position,* $x = 0$*, thus* $\boldsymbol{v_{max}} = \boldsymbol{\omega A}$ *. Thus* v *is maximum at mean position and zero at extreme position.*

3. Expression for the acceleration of the particle executing SHM

The displacement of the particle is given by $x = A\, sin\,(\omega t + \varphi)$. If the particle starts from mean position when t = 0, then initial phase $\varphi = 0$. Thus the above equation is

$x = A\, sin\omega t$. Differentiating $\frac{dx}{dt} = A\omega\, cos\omega t$

Differentiating again, we get $\frac{d^2x}{dt^2} = -A\omega^2\, sin\omega t = -A\omega^2\left(\frac{x}{A}\right)$ (as $sin\omega t = \frac{x}{A}$)

or the acceleration $\frac{d^2x}{dt^2} = \boldsymbol{a = -\omega^2 x}$. This gives the expression for the acceleration of the particle executing SHM.

Note : *At the mean position, when x = 0, a = 0. Also at the extreme position x = A, then* $a_{max} = -\omega^2 A$ *i.e. a is maximum at the extreme position.*

4. Expression for time period of a particle executing SHM

The angular frequency of the particle executing SHM is given by $\omega = 2\pi f$ where f is the frequency. As $f = \frac{1}{T}$, where T is the time period of SHM, we get $\omega = \frac{2\pi}{T}$ or $T = \frac{2\pi}{\omega}$.

This equation can be written as $T = 2\pi\sqrt{\frac{1}{\omega^2}}$ . .....(1) As $a = \omega^x x$, $\omega^2 = \frac{a}{x}$.

The equation (1) is $\boldsymbol{T = 2\pi\sqrt{\frac{x}{a}}}$. Thus $T = 2\pi\sqrt{\frac{displacement}{acceleration}}$.

As $a = \left(\frac{k}{m}\right)x$, and $a = \omega^x x$, Thus $\omega^2 = \frac{k}{m}$, leading to $\boldsymbol{T = 2\pi\sqrt{\frac{m}{k}}}$

ot $T = 2\pi\sqrt{\frac{mass}{force\ per\ unit\ displacement}}$.

5. Expression for energy of a particle executing SHM

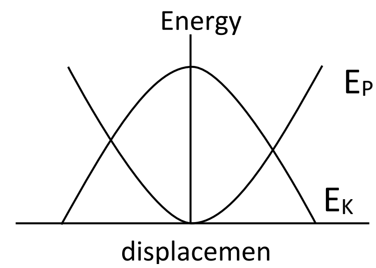
Consider a particle executing SHM represented by $y = A\, sin\,(\omega t + \varphi)$. The velocity of the particle is $\upsilon = \omega\sqrt{A^2 - \omega^2}$ and its acceleration is $a = -\omega^2 x$.

The energy of the particle is the sum of its kinetic and potential energies.

The particle having a displacement x is further moved through a small distance dx, then the work done ***dW = - force × distance moved***. The negative sign indicates that work is done against the direction of restoring force.

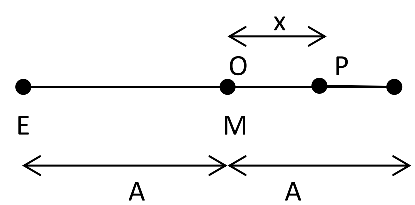
Thus the work done is given by $dW = -F \times dx = -ma \times dx$.

As $a = -\omega^2 x$, the work done is $dW = -m(-\omega^2 x) \times dx$
or $dW = m\omega^2 x\, dx$
Total work done in moving the particle from 0 to x is given by integrating the above equation, $\int dW = \int_0^x m\omega^2 x\, dx$ or $\boldsymbol{E_P = W = \frac{1}{2} m\omega^2 x^2}$ .....(1)
This expression gives the **potential energy E_P** of the particle.



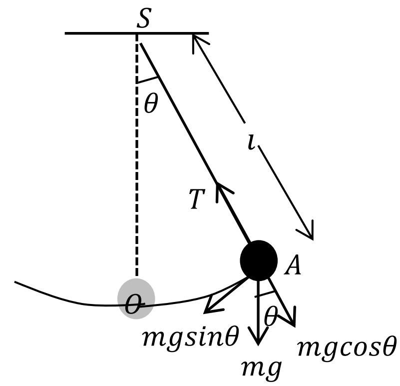
The kinetic energy of the particle is $E_K = \frac{1}{2} mv^2$.
As $v = \omega\sqrt{A^2 - x^2}$,
Kinetic energy is given by $\boldsymbol{E_K = \frac{1}{2} m\omega^2 (A^2 - x^2)}$ ....(2)
The total energy of the particle is $E = E_P + E_K$ *or*
$E = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 (A^2 - x^2)$
Thus $\boldsymbol{E = \frac{1}{2} m\omega^2 A^2}$ ...(3) or $E = \frac{1}{2} m(2\pi f)^2 A^2$ or $\boldsymbol{E = 2\pi^2 m f^2 A^2}$
This is the expression for the energy of the particle executing SHM.

<u>To show that the total energy of the oscillating particle is the same at every point of oscillation</u>



The expression for kinetic energy is $E_K = \frac{1}{2} m\omega^2 (A^2 - x^2)$. ...(1)
and potential energy expression is $E_p = \frac{1}{2} m\omega^2 x^2$ ....(2)
The total energy at any point of oscillation is
$E = E_P + E_K = \frac{1}{2} m\omega^2 A^2$ ....(3)
1 : At the extreme position, $x = A$ and thus from (1) kinetic energy is zero (as the velocity is also zero). Thus $E_K = 0$.
Also from (2) As $x = A$, at the extreme position, potential energy is $E_p = \frac{1}{2} m\omega^2 A^2$.
Total energy $E = E_P + E_K = \frac{1}{2} m\omega^2 A^2$ .....(4)
2 : At the mean position, kinetic energy is maximum as the velocity is maximum. Also as $x = 0$ at mean position, from (1) $E_K = \frac{1}{2} m\omega^2 A^2$.
and as $x = 0$, at the mean position, from (2) $E_p = 0$. Thus total energy
$E = E_P + E_K = \frac{1}{2} m\omega^2 A^2$ .....(5) Thus from expressions (3), (4) and (5) it is clear that total energy remains same at every point of oscillation.

<u>Simple pendulum- Expression for time period :</u> A heavy particle suspended by means of a light inelastic, torsion less thread is called **simple pendulum**.
When the pendulum is pulled to one side and let go, it oscillates in a vertical plane under gravity. The oscillations are simple harmonic.



Consider a simple pendulum of length l, oscillating about the vertical plane as shown. Let θ be the angular displacement.

The forces acting on the pendulum are **(i) tension T** of the string acting along AS and **(ii) weight W = mg** acting vertically downwards.

The weight mg is resolved into two components, (i) along the string i.e. $mgcos\theta$ and the (ii) perpendicular to it i.e. $F = -mgsin\theta$. The second one is the restoring force.

The acceleration by definition is the ratio of force to the mass.

Thus $a = \frac{F}{m} = -\frac{mgsin\theta}{m} = -gsin\theta$

As the amplitude of oscillation is small, $sin\theta$ can be approximated to θ .

Thus $a = -g\,\theta$ and it is directed towards the mean position.

From the diagram $\theta = \frac{OA}{l} = \frac{x}{l}$ where OA = x is the linear displacement of the pendulum.

Thus $a = -g\frac{x}{l} = -\left(\frac{g}{l}\right)x$ …..(1) or $a \propto x$.

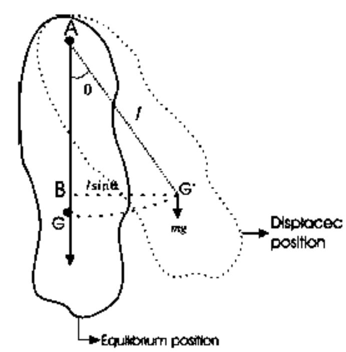
This shows that oscillations of pendulum are simple harmonic.

The time period of oscillation is given by $T = 2\pi\sqrt{\frac{x}{a}}$ ……..(2)

Substituting for a from (1) in (2) (neglecting -ve sign) $T = 2\pi\sqrt{\frac{x}{\left(\frac{g}{l}\right)x}}$ or $\boldsymbol{T = 2\pi\sqrt{\frac{l}{g}}}$

This is the expression for the time period of oscillations of a simple pendulum.

<u>Compound pendulum – Expression for time period</u>



A rigid body capable of oscillating freely about a horizontal axis in a vertical plane is called **compound pendulum.** The point where the axis of rotation intersects the vertical plane of the pendulum through the centre of gravity is called the **centre of suspension**.

Consider an extended body as shown. Let A be the point of suspension and G the centre of mass (centre of gravity). Let θ be the angle subtended between the downward vertical (which passes through point) and the line AG'.

At this position, the weight of the object acts downwards. Thus, the moment of the couple which tends to restore the body to equilibrium position ($\theta = 0$) is given by

$\tau = -mg\,l\,sin\theta$. ……(1) (*moment of the couple = Force × perpendicular distance = mg × BG' and BG' = l sinθ from the right angle triangle ABG'*)

As θ is small $sin\theta \approx \theta$. Thus $\tau = -mg\,l\,\theta$ …..(2)

Its angular acceleration is $\alpha = \frac{d^2\theta}{dt^2}$.

The moment of the couple is $\tau = I\,\alpha = I\,\frac{d^2\theta}{dt^2}$ .....(3) where I is the moment of inertia.

Comparing equations (2) and (3) , $I\,\frac{d^2\theta}{dt^2} = -\,mg\,l\,\theta$ or $\frac{d^2\theta}{dt^2} = -\frac{mg\,l}{I}\,\theta$

or $\boldsymbol{\alpha = \frac{d^2\theta}{dt^2} = -\left(\frac{mg\,l}{I}\right)\theta}$ .....(4) or $\alpha \propto \theta$. Thus angular acceleration is directly proportional to angular displacement. Thus the oscillations of a compound pendulum is simple harmonic.

The time period of oscillations is given by $T = 2\pi\sqrt{\frac{angular\ displacement}{angular\ acceleration}} = 2\pi\sqrt{\frac{\theta}{\alpha}}$

Substituting for α from equation (4) in the above equation, we get (neglecting negative sign) $T = 2\pi\sqrt{\frac{\theta}{\left(\frac{mg\,l}{I}\right)\theta}}$ or $\boldsymbol{T = 2\pi\sqrt{\frac{I}{mg\,l}}}$ ...... (5)

From the parallel axis theorem $I = I_g + ml^2$ where I_g is the moment of inertia about an axis through the centre of gravity G. It is given by $I_g = mK^2$ where K is the radius of gyration. Thus the above equation becomes

$I = mK^2 + ml^2 = m\,(K^2 + l^2)$ ....(6)

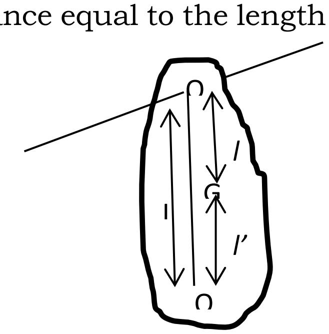
Substituting for I from (6) in (5), we get $T = 2\pi\sqrt{\frac{m\,(K^2+l^2)}{mg\,l}} = 2\pi\sqrt{\frac{(K^2+l^2)}{g\,l}}$

or $T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}}$ or $\boldsymbol{T = 2\pi\sqrt{\frac{L}{g}}}$ where $L = \frac{K^2}{l} + l$

This is the expression for time period of oscillations of a compound pendulum.
This expression is identical in form to the corresponding expression for a simple pendulum. Thus, a compound pendulum behaves like a simple pendulum with *effective length* L called length of equivalent simple pendulum.

<u>To show that the centre of oscillation and the centre of suspension are interchangeable</u>

Let point O be the centre of suspension. The point O_1 which is at a distance equal to the length equivalent simple pendulum from the centre of suspension along OG is called the centre of oscillation. $OO_1 = L$. Let $OG = l$ *and* $GO_1 = l'$.
Let the pendulum be suspended about O. If T is the period of oscillation



about O, then $T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}}$

But $\frac{K^2}{l} + l = L = l + l'$. Thus $l' = \frac{K^2}{l}$ or $K^2 = ll'$

Substituting for k² in the expression for T, we get $T = 2\pi\sqrt{\frac{\frac{ll'}{l}+l}{g}} = 2\pi\sqrt{\frac{l'+l}{g}}$

Thus $T = 2\pi\sqrt{\frac{L}{g}}$ . ...(1) This is because $L = l + l'$

Let the pendulum be suspended about O_1, then the period of oscillation about O_1 is

$T' = 2\pi\sqrt{\frac{\frac{K^2}{l'}+l'}{g}}$. But $\frac{K^2}{l'} + l' = L = l + l'$. Thus $l = \frac{K^2}{l'}$ or $K^2 = ll'$

Substituting for k² in the expression for T', we get $T' = 2\pi\sqrt{\frac{\frac{ll'}{l'}+l'}{g}} = 2\pi\sqrt{\frac{l+l'}{g}}$

Thus $T' = 2\pi\sqrt{\frac{L}{g}}$ . ...(2) This is because $L = l + l'$

Comparison of equations (1) and (2) shows that the period is same when the pendulum is suspended about either O or O_1.

<u>Condition for maximum and minimum period of oscillation</u>

The period of oscillation of the compound pendulum is given by $T = 2\pi\sqrt{\frac{(K^2+l^2)}{g\,l}}$ ...(1)

Squaring the above equation, we get $T^2 = 4\pi^2$ we get $\frac{(K^2+l^2)}{g\,l} = 4\pi^2\frac{\left(\frac{k^2}{l} + l\right)}{g}$

Differentiating with respect to l , $2T\frac{dT}{dl} = \frac{4\pi^2}{g}\left(-\frac{k^2}{l^2} + 1\right)$ ....(1)

For maximum or minimum $\frac{dT}{dl} = 0$, Hence $-\frac{k^2}{l^2} + 1 = 0$ or $k^2 = l^2$ i.e. $l = \pm k$

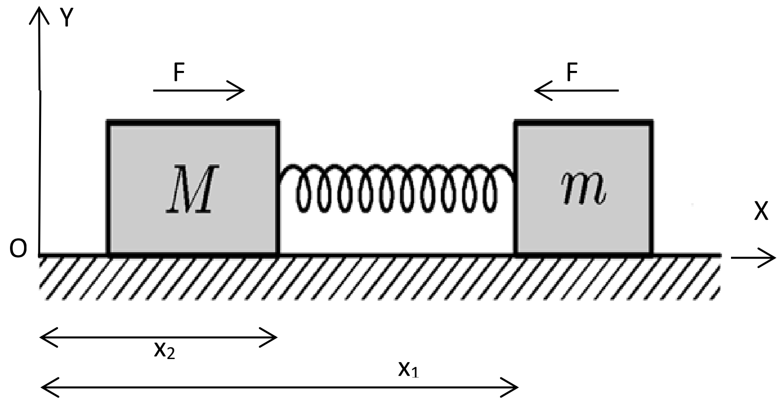
Equation (1) can be written as $T = 2\pi\sqrt{\frac{(k-l)^2-2kl}{g\,l}}$ The condition for minimum is $l = k$

Putting this condition in the above equation, $T_{min} = 2\pi\sqrt{\frac{2k}{g}}$. This is the expression for minimum period. When point of suspension passes through the centre of gravity G then $l = 0$,

Putting this condition in equation $T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}}$ period T becomes infinite and thus period T **is maximum**. Hence the period of oscillation becomes maximum when the axis of suspension passes through the centre of gravity.

<u>Oscillations of two masses connected by a spring:</u> Consider two masses M and m connected by a light spring as shown. The masses are slightly pulled in opposite direction so that they oscillate simple harmonically along the X axis.

Let the equilibrium length of the spring be l . If x_1 and x_2 are the coordinates of the ends of the spring and let x be the extension of the spring when it oscillates,



then $x = (x_1 - x_2) - l$

or $(x_1 - x_2) = x \mp l$

The differential form of SHM for the two masses are $m\frac{d^2x_1}{dt^2} = -kx$ .....(1)

and $M\frac{d^2x_2}{dt^2} = +kx$ .....(2)

Multiplying (1) by M and (2) by m and subtracting, we get

$Mm\frac{d^2x_1}{dt^2} - Mm\frac{d^2x_1}{dt^2} = -Mkx - (mkx)$

or $Mm\left(\frac{d^2x_1}{dt^2} - \frac{d^2x_1}{dt^2}\right) = -k(M+m)x$

or $\frac{d^2}{dt^2}(x_1 - x_1) = -k\left(\frac{m+M}{mM}\right)x$ or $\frac{d^2x}{dt^2} = -k\left(\frac{m+M}{mM}\right)x$

where $\frac{d^2}{dt^2}(x_1 - x_1) = \frac{d^2x}{dt^2}$ as l is a constant and putting $\frac{mM}{m+M} = \mu$ called reduced mass of the system, we get $\frac{d^2x}{dt^2} = -\frac{k}{\mu}x$ or $\frac{d^2x}{dt^2} = -\omega^2 x$

Thus $\omega^2 = \frac{k}{\mu}$. The time period of the oscillator is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1}{\omega^2}}$

thus $\boldsymbol{T = 2\pi\sqrt{\frac{\mu}{k}}}$. This is same as the time period of a single body oscillator.

Damped oscillations

In real oscillating systems, mechanical energy is lost from the system due to frictional or any other dissipative forces leading to decrease in amplitude and finally oscillations stop.

If the amplitude of the oscillations of a system decreases with time then it is called **damped oscillations**. If the oscillations of a system persist without any change in its amplitude then it is called **undamped oscillations.** Three possible situations arise. They are

1. When the amplitude of the oscillation decreases very slowly, the system is said to be **underdamped**. Eg., when a tuning fork is set into vibrating, the sound of vibrations persists for quite sometime before stopping.
2. When non-vibratory motion of a system occurs in the shortest time interval, the system that comes to equilibrium position very quickly is said to be **critically damped**. Eg. spring system in a moving coil meter and in digital mass measurement device is critically damped.
3. When non-vibratory motion of a system occurs and it takes a long time for the system to come to rest to its equilibrium position, the system is said to be **overdamped**. Eg. Heavy public doors on some building are overdamped to prevent them closing too quickly, giving time for people to enter and so that the

doors are do not close quickly. The doors have some hydraulic dashpot (type of shock absorber) to provide the damping.

Equation for damped simple harmonic oscillation

Consider a body executing damped harmonic oscillations. Let y be the displacement of the body from its mean position at an instant of time t.

Let $v = \frac{dy}{dt}$ be its instantaneous velocity.

Two forces are acting on the oscillating body. They are

(i) The restoring force which acts opposite to the direction of displacement and is given by $F_r = -ky$ where k is the constant of proportionality.

(ii) The frictional force which acts opposite to the direction of velocity of the body and is given by $F_f = -bv = -b\frac{dy}{dt}$ where b is the constant of proportionality called damping coefficient.

The total force acting on the body is $F = F_r + F_f = -ky - bv = -ky - b\frac{dy}{dt}$

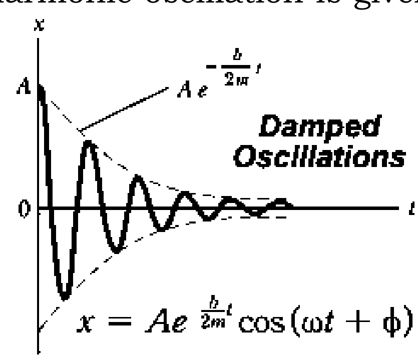
From Newton's second law $F = ma = m\frac{d^2y}{dt^2}$.

The above equation becomes $m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt}$

or $$\boldsymbol{\frac{d^2y}{dt^2} + \frac{k}{m}y + \frac{b}{m}\frac{dy}{dt} = 0} \quad \ldots\ldots(1)$$

Equation (1) is the differential form of damped simple harmonic oscillations. The solution of the above equation is of the form $y = Ae^{-\left(\frac{b}{2m}\right)t}\cos(\omega' t + \theta)$ where $\frac{b}{m} = \frac{1}{\tau}$ with τ called the relaxation time.

The expression for the angular frequency of the damped harmonic oscillation is given by $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ and the amplitude of the oscillation is $Ae^{-\left(\frac{b}{2m}\right)t}$.

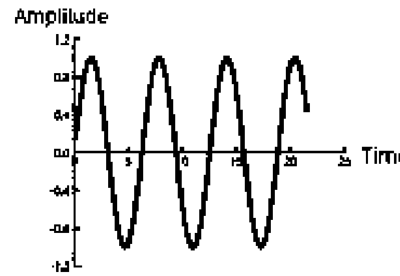


Thus the amplitude decreases with time which depends on the value of b.

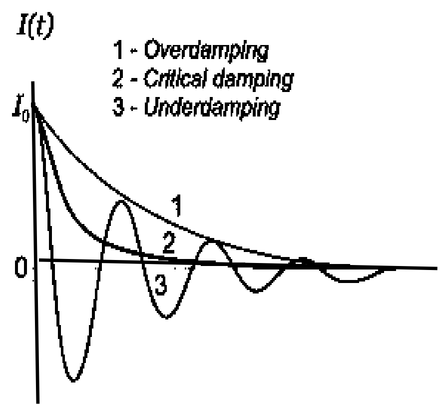
As SHM is represented by a sine function, the damped oscillation is represented by sine wave of decreasing amplitude as shown in the diagram. If the amplitude of oscillations remains constant as shown then oscillations are said to be undamped represented by sine wave of constant amplitude as shown. The expression for angular frequency in this case is $\omega' = \sqrt{\frac{k}{m}}$.

Case 1 Condition for Overdamping :

When $\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$, $\boldsymbol{b > \sqrt{4km}}$, the condition is called overdamping. Under this condition, the body that is displaced from equilibrium position comes back to its initial position slowly without oscillating. Here b is large. There are no oscillations, but the decay can be quite slow because the friction is so high that it's hard for the mass to move (curve 1).



Undamped System



Case 2 Condition for Critically damping – When $\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$ This is the condition for critically damped case. Here the body comes back to the initial state more rapidly than the over damped condition. Here b is optimum where the body does not oscillate but comes to mean position quickly (curve 2).

Case 3 Condition for Underdamping – When $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$, the condition is called under damping. In this case the body continues to oscillate with decreasing amplitude and finally comes to initial state (curve 3).

Forced oscillations When a body that is capable of oscillating with its natural frequency is subjected to an external periodic force or the driving force, it oscillates under the action of applied periodic force. Such oscillations are called forced oscillations.

Equation of simple harmonic oscillations under the action of periodic force

Consider a body of mass m oscillating under the action of an external periodic force of angular frequency ω_p . The periodic force can be $F_p = F_0 \, sin\omega_p t$ where F_0 is the maximum value of this force. The body is experiencing restoring force $F_r = -ky$ and frictional force $F_f = -bv$ along with the periodic force.

If y is the displacement of the body and $\frac{dy}{dt}$ is its velocity at an instant of time t , then

$$F = F_r + F_f + F_p = -ky - bv + F_0 \, sin\omega_p t = -ky - b\frac{dy}{dt} + F_0 \, sin\omega_p t$$

From Newton's second law $F = ma = m\frac{d^2y}{dt^2}$.

Thus the above equation becomes $m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F_0 \, sin\omega_p t$

The above equation can be written as $\frac{d^2y}{dt^2}+\frac{k}{m}y+\frac{b}{m}\frac{dy}{dt}=\frac{F_0}{m}\,sin\omega_p t$ ……(1)

Putting $\frac{k}{m}=\omega^2$, $\frac{b}{m}=\frac{1}{\tau}$ and $\frac{F_0}{m}=f$, τ is the relaxation time, equation (1) can be written as $\boldsymbol{\frac{d^2y}{dt^2}+\omega^2\,y+\frac{1}{\tau}\frac{dy}{dt}=fsin\omega_p t}$ …..(2)

This is the differential equation of damped and forced simple oscillations.

The solution of equation (2) is $y=A\,sin(\omega_p t+\theta)$ …..(3) where A is the amplitude of oscillations and θ is phase angle by which the displacement y lags behind the applied periodic force. Differentiating equation (3) with respect to t , $\frac{dy}{dt}=A\omega_p\,cos(\omega_p t+\theta)$ and differentiating again $\frac{d^2y}{dt^2}=-A\omega_p^2\,sin(\omega_p t+\theta)$

Substituting the values of y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in equation (2), we get

$-A\omega_p^2\,sin(\omega_p t+\theta)+\omega^2 A\,sin(\omega_p t+\theta)+\frac{1}{\tau}A\omega_p\,cos(\omega_p t+\theta)=fsin\omega_p t$ or

$-A\omega_p^2\,sin(\omega_p t+\theta)+\omega^2 A\,sin(\omega_p t+\theta)+\frac{1}{\tau}A\omega_p\,cos(\omega_p t+\theta)=fsin[(\omega_p t+\theta)-\theta]$

$-A\omega_p^2\,sin(\omega_p t+\theta)+\omega^2 A\,sin(\omega_p t+\theta)+\frac{1}{\tau}A\omega_p\,cos(\omega_p t+\theta)=f\,[sin(\omega_p t+\theta)\,cos\theta-cos(\omega_p t+\theta)sin\theta]$

Comparing the coefficients of $sin(\omega_p t+\theta)$ and $cos(\omega_p t+\theta)$ on either side of the above equation, we get $-A\omega_p^2+\omega^2 A=f\,cos\theta$ or $A(\omega^2-\omega_p^2)=f\,cos\theta$ ….(4)

and $\frac{1}{\tau}A\omega_p=f\,sin\theta$ ….(5)

squaring and adding equations (4) and (5), we get $A^2(\omega^2-\omega_p^2)^2+\frac{A^2\omega_p^2}{\tau^2}=f^2$

This equation can be written as $A^2=\frac{f^2}{(\omega^2-\omega_p^2)^2+\frac{\omega_p^2}{\tau^2}}$ or $A^2=\frac{(F_0/m)^2}{(\omega^2-\omega_p^2)^2+\frac{b^2\omega_p^2}{m^2}}$

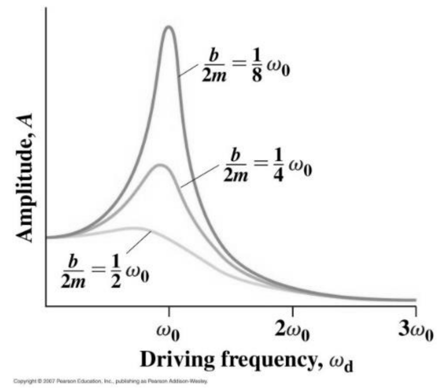
Thus the amplitude of oscillations is given by $\boldsymbol{A=\frac{F_0/m}{\sqrt{(\omega^2-\omega_p^2)^2+\frac{b^2\omega_p^2}{m^2}}}}$ …(6)

The amplitude of oscillation is directly proportional to the amplitude of the periodic force F_0 . The amplitude also depends on the difference between the applied frequency and the natural frequency of oscillation of the body.

Resonance When the body is oscillating with the frequency of the applied periodic force equal to the natural frequency of the body, then the oscillations are said to be in resonance. Under this condition of $\omega=\omega_p$, the amplitude of oscillation is maximum i.e. $A=A_{max}$.

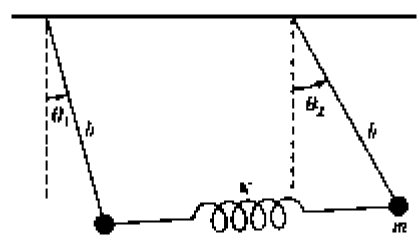
Putting this condition in the equation $A = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_p^2)^2 + \frac{b^2\omega_p^2}{m^2}}}$ we get

$A_{max} = \frac{F_0/m}{\sqrt{\frac{b^2\omega_p^2}{m^2}}} = \frac{F_0}{b\,\omega_p}$. When $b \to 0$, then $A_{max} \to \infty$.



This indicates that for zero damping the amplitude of oscillation is infinity. As the damping coefficient increases, the amplitude of oscillation decreases as shown in the diagram. In the diagram $\omega_0 = \omega$ called the resonant frequency. The driving force is the periodic force.

<u>Examples of resonance</u> – 1. *When you hear annoying buzzing sounds in your car, it is often something ``resonating" with the car engine. 2. When a child, or adult, is on a swing, they can increase their amplitude of oscillation by ``pumping" the swing at just the right moments. 3. A microwave oven works on principle of resonance. The microwaves apply periodic forces to water molecules causing them to resonate, and hence heat up. A huge bridge may collapse right after it was built. That is an example of resonance. The wind was applying forces at just the right frequency to get the whole thing to swing wildly back and forth, ultimately collapsing.*



<u>Coupled oscillations :</u> When a number of oscillating systems are connected in such a manner that the energy is exchanged between them, then the oscillators are said to be coupled.
Consider two simple pendulums A and B whose bobs are connected by a spring as shown. The natural length of the spring is equal to the distance between the pendulum bobs in their equilibrium positions. This arrangement is called a coupled oscillator and the oscillations produced by them is simple harmonic and are called **coupled oscillations**.
This system has two normal modes of oscillations. (i) In phase mode and (ii) out of phase mode. When the bobs of the two pendulums are displaced equally in the same direction and allowed to oscillate, it is referred to as in phase mode. As the coupling spring is in its relaxed length, the spring does not exert any force on the pendulums. Thus each pendulum oscillates with its natural frequency independent of the other. The restoring force is given by $F = -m\omega^2 x$. The angular frequency of oscillation is given by $\omega_0 = \sqrt{\frac{g}{l}}$. The displacements are represented by $x_1 = A\,cos\omega_0 t$ and $x_2 = A\,cos\omega_0 t$.
When the bobs of the two pendulums are displaced equally in opposite directions and allowed to oscillate, it is called out of phase mode. In this case the coupling spring undergoes stretching and compression alternately exerting force on both the pendulums. In this case the coupling

spring exerts a restoring force given by $F = -2kx$ where k is the spring constant. Thus the equation of motion for pendulum A is given by $m\frac{d^2x_1}{dt^2} = -\mathrm{m}\omega_0^2 x_1 - 2\mathrm{k}x_1$. This is the differential equation. This equation can be written as

$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 + 2\frac{k}{m}x_1 = 0$ *or* $\frac{d^2x_1}{dt^2} + (\omega_0^2 + 2\omega_c^2)x_1 = 0$ where $\omega_c^2 = \frac{k}{m}$ angular frequency of the coupling spring. The angular frequency is $\omega' = \sqrt{(\omega_0^2 + 2\omega_c^2)} = \sqrt{\frac{g}{l} + \frac{2k}{m}}$

The solution of the above differential equation is $x_1 = C\, cos\omega' t$. Similarly the expressions for the displacement of pendulum B can be arrived at. As they are in out of phase the equation is $x_2 = C\, cos\omega' t$ .The coupling in this case has increased the angular frequency of oscillations.

Energy transfer in coupled oscillator

The effect of coupling between the two oscillating systems is explained in terms of the energy transfer between them. Keeping one of the bobs at rest, other bob is displaced from the equilibrium position and is set into oscillations. The bob which is at rest picks up the oscillations and the displaced bob comes to rest. This is due to the energy transfer between the coupled pendulums. One cycle of transfer of energy is between two successive positions of a bob coming to rest. It can be shown that the angular frequency of the energy transfer is given by $\omega' - \omega_0$ where ω' is the frequency of coupled oscillator in out of phase mode and ω_0 is the frequency when it is in phase mode. This difference is called beat frequency and the reciprocal of this is the beat period of oscillation.

PART A : Descriptive questions (8 mark each)

1 (a) Define periodic motion giving an example.
(b) Obtain differential equation for a simple harmonic motion and write down the expression for the angular velocity and time period of simple harmonic motion.

2. (a) Define phase of a particle executing SHM. What is epoch?
(b) Derive differential equation for simple harmonic motion and hence arrive at the expression for the velocity and acceleration of the particle executing simple harmonic motion.

3. (a) Derive an expression for the energy of a particle executing simple harmonic motion. Represent the variation of energy graphically.
(b) Show that total energy of simple harmonic motion remains constant for all values of displacement.

4. (a) What is time period of oscillation?
(b) Derive an expression for time period of small amplitude of a simple pendulum. What happens to the time period of its oscillations if the amplitude is large?

5. (a) What is compound pendulum? Derive expressions for its time period and reduced length.
(b) Show that the centre of suspension and centre of oscillation of a compound pendulum are interchangeable.

6. (a) What are damped oscillations? Explain.
(b) Arrive at the differential equation for damped oscillations and mention the conditions for underdamped, critically damped and over damped cases.

7. (a) What are forced oscillations? Explain.
   (b) Arrive at the expression for the amplitude of forced oscillations.
8. (a) With the help of the expression for amplitude of forced oscillations discuss the condition for resonance.
   (b) What are coupled oscillations? Explain the energy transfer in coupled oscillations.
9 (a) Define time period and frequency of SHM.
   (b)Arrive at the expression for time period of two masses connected by a spring.

PART B : <u>Numerical problems (4 mark each)</u>

1. A particle executes simple harmonic motion of period $22\ s$ and amplitude $0.1\ m$. Find the distance of the particle from the mean position in $3.5\ s$.
   [*Hint :* $\omega = \frac{2\pi}{T}$ *and* $x = A\sin\omega t$ *Ans :* $x = 0.8416\ m$]
2. An object of mass 0.05 kg is executing SHM. A force of 0.5 N is acting on it when the displacement from the mean position is 0.1 m. Find the time period if the maximum velocity is 5 ms^{-1}. Calculate the amplitude and maximum acceleration. Also calculate its kinetic and potential energies when the displacement is 0.1 m. Also find the total energy.
   [Hint : $k = \frac{F}{x}$, $T = 2\pi\sqrt{\frac{m}{k}} = 0.628\ s$ $\omega = \frac{2\pi}{T}$, $A = \frac{v_{max}}{\omega} = 0.5\ m$, $a_{max} = \omega^2 A = 50\ ms^{-2}$
   $E_K = \frac{1}{2}\ m\omega^2(A^2 - x^2) = 0.6\ J$, $E_P = \frac{1}{2}\ m\omega^2 x^2 = 0.025\ J$, $E_T = \frac{1}{2}\ m\omega^2 A^2 = 0.625\ J$]
3. A particle executing SHM makes seven oscillations in 11 seconds. The velocity of the particle is 1.2 ms^{-1} when its distance from the centre of oscillation is 12.5 cm. Find the amplitude of motion, the maximum velocity and maximum acceleration.
   [Hint : $T = \frac{11}{7}\ s$, $\omega = \frac{2\pi}{T}$, $v = \omega\sqrt{A^2 - x^2}$, $A = 0.464\ m$, $v_{max} = \omega A = 1.856\ ms^{-1}$, $a_{max} = \omega^2 A = 7.424\ ms^{-2}$]
4. A particle executing SHM makes 25 oscillations per minute. The speed of the particle in its mean position is 12 ms^{-1}. Find the velocity when the particle is midway between the mean position and one extreme position.
   [Hint : $T = \frac{60}{25}\ s$, $\omega = \frac{2\pi}{T}$, $v_{max} = v_{mean} = \omega A$, $A = \frac{v_{max}}{\omega} = 4.585\ m$, $To\ find\ v = \omega\sqrt{A^2 - x^2}\ when\ x = \frac{A}{2}$, $v = 10.37 ms^{-1}$
5. A spring whose force constant is 100 Nm^{-1} hangs vertically supporting a 2 kg mass at rest. Find the distance by which the mass should be pulled down so that on being released it may pass the equilibrium position with a velocity of 1 ms^{-1}.
   [Hint : $T = 2\pi\sqrt{\frac{m}{k}}$, $v_{max} = \omega A$, $distance\ A = 0.13\ m$]
6 If the potential energy of a particle performing SHM is 2.5 J, when displacement is half of amplitude, find the total energy.
   [Hint : $E_P = \frac{1}{2}\ m\omega^2 x^2$, $E_T = \frac{1}{2}\ m\omega^2 A^2$ $\frac{E_T}{E_P} = \frac{A^2}{x^2}$ $where\ x = \frac{A}{2}$, $E_T = 10\ J$]
7 An oscillator consists of a block of mass 512 g connected to a spring. When set into oscillation with amplitude 34.7 cm, it is observed to repeat its motion every 0.484s. Find (a) angular

frequency, (b) force constant, (c) maximum speed and (d) the maximum force exerted on the block.

[Hint : $T = 0.484$, $\omega = \frac{2\pi}{T} = 12.97\, rads^{-1}, \omega^2 = \frac{k}{m}$ or $k = m\omega^2 = 86.13\, Nm^{-1}$, $v = \omega A = 4.5\, ms^{-1}, F_{max} = kA\ (as\ x = A)$, $F_{max} = 29.9\ N$]

8 A body is executing SHM with an amplitude of 0.15 m and frequency 4Hz. commute (a) maximum velocity of the body (b) acceleration when displacement is 0.09m and time required to move from mean position to a point 0,12 m away from it.

[Hint : $\omega = 2\pi f$, $v_{max} = \omega A = 3.768\, ms^{-1}$, $a = \omega^2 x = 56.8\, ms^{-2}, x = A\ sin\omega t\ , t = 0.0734\ s$]

9 A body of mass M attached to a spring is oscillating with time period 2s. If the mass of the body is increased by 2 kg, its time period increases by 1s. If Hooke's law holds good, calculate the initial mass.

[Hint : $T = 2\pi\sqrt{\frac{m}{k}}$, $T^2 = 4\pi^2\frac{m}{k}$, $\frac{T_2^2}{T_1^2} = \frac{m+2}{m}$, $m = 1.6\ kg$]

10 The pan attached to a spring balance has a mass of 1 kg. A mass of 2 kg when placed on the pan stretches the spring by 10 cm. what is the frequency with which the empty pan will oscillate?

[Hint : $F = kx$ $and\ F = mg$, $k = \frac{mg}{x} = \frac{(1+2)\ 9.8}{0.1} = 294 Nm^{-1}$, $\omega^2 = \frac{k}{m'}$ $and\ \omega = 2\pi f$, $4\pi^2 f^2 = \frac{k}{m'}$ or $f = \frac{1}{2\pi}\sqrt{\frac{k}{m'}} = = 2.73\ Hz$ $(Here\ m' = 1\ kg)$]

11. A simple pendulum has a period of 4 seconds. If the length of the pendulum is shortened by 1 m the period gets reduced by 0.56 seconds. Calculate the acceleration due to gravity.

[Hint : $T = 2\pi\sqrt{\frac{l}{g}}$ $T_1 = 2\pi\sqrt{\frac{l}{9.8}}$, $l = 3.97\ m$ and $l' = 3.97 - 1 = 2.97\ m$, $T_2 = 4 - 0.56 = 3.44\ s$, $g = 4\pi^2\frac{l'}{T_2^2} = 9.898\ ms^{-2}$]

12 Two simple pendulums of length $1.44\ m$ and $1\ m$ start swinging at the same time. Calculate the ratio of their time periods.

[Hint : $T = 2\pi\sqrt{\frac{l}{g}}$, $T \propto \sqrt{l}\ , thus$ $\frac{T_1}{T_2} = \sqrt{\frac{l_1}{l_2}} = 1.2 : 1$]

13 A thin uniform bar of length 1.2 m is made to oscillate about an axis through its end. Find the prod of oscillation and other points about which it can oscillate with the same period.

[Hint : Moment of inertia of the rod is $I = \frac{Ml^2}{12} = MK^2$, $Thus\ K = \frac{l}{\sqrt{12}}$, $T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}} = 1.8s$ $\left(l = \frac{1.2}{2}\right) Theother\ point\ is\ at\ distance\ L,\ thus\ T = 2\pi\sqrt{\frac{L}{g}}$ or $L = 0.8\ m$]

14 A uniform rod $1.5\ m$ in length oscillates about a horizontal axis at one end, perpendicular to its length. (a) Find the position of a point about which the time period is minimum. (b) Find the minimum time period $g = 9.8\ ms^{-2}$. [Hint: $I = \frac{Ml^2}{12} = MK^2$, $Thus\ K = \frac{l}{\sqrt{12}} = \frac{1.5}{\sqrt{12}} = 0.433\ m$, $For\ min.period\ l = k = 0.433m$, $T = 2\pi\sqrt{\frac{2K}{g}} = 1.866\ s$

15 The time period of a bar pendulum is 1.53 second when the centre of suspension is 0.3 m from one end and 1.49 second when it is 0.2 m from the same end. If the bar is 1 m long, find the acceleration due to gravity.

[Hint : $l_1 = 0.5 - 0.3 = 0.2m$ *and* $l_2 = 0.5 - 02 = 0.3m$ *where* $0.5m$ *is distance of centre of gravity to one end as length of bar is* $1m$*. As* $L = \frac{K^2}{l} + l$ *and* $T \propto \sqrt{L}$ *thus* $\frac{T_1}{T_2} = \sqrt{\frac{L_1}{L_2}}$ *find* $\frac{L_1}{L_2}$, *and* $\frac{L_1}{L_2} = \left(\frac{K^2}{l_1} + l_1\right)\frac{}{\frac{K^2}{l_2}} + l_2$ *find* K *and* $g = 4\pi^2 \frac{L_1}{T_1^2} = 9.85\ ms^{-2}$]

16 A uniform metre stick is suspended through a small pin hole at the 10 cm mark. Find the time period of small oscillation about the point of suspension.

[Hint : Metre stick is 1m long, distance of centre of gravity to point of suspension is 0.4 m thus $l = 0.4m$ $\quad I = \frac{Ml^2}{12} = MK^2$, *Thus* $K = \frac{l}{\sqrt{12}} = 0.115m$, $T = 2\pi\sqrt{\frac{\frac{K^2}{l}+l}{g}} = 1.32\ s$

PART C : <u>Conceptual questions (2 mark each)</u>

1. What happens to the time period of a simple harmonic motion when its amplitude is doubled? Explain.
   Ans : Amplitude of SHM is independent of time period. Thus no change in time period.
2. All simple harmonic motions are periodic, but all periodic motions are not simple harmonic motions. Explain.
   Ans : Periodic motion can be both revolutionary or rotatory motion and oscillatory motion. Thus all SHMs are be periodic but all periodic motion need not be simple harmonic.
3. Can a body have zero velocity but still a varying acceleration? Explain.
   Ans : During an oscillatory motion, at the extreme position of oscillation, velocity becomes zero when the body stops momentarily but the acceleration acts to bring it back due to restoring force.
4. What provides the restoring force for SHM in the case of simple pendulum and a loaded spring?
   Ans : It is the force of gravity (its component) which is the restoring force in case of simple pendulum. It is the deforming force which is the elastic force in the spring responsible for restoring force in case of loaded spring.
5. What is the time period of a simple pendulum in a satellite? Explain
   Ans : As there is no gravity in a satellite, the time period of a simple pendulum is infinity.
6. At what distance from the mean position is the kinetic energy equal to the potential energy of the oscillating body?
   Ans : When the kinetic energy and potential energy are equal, $\frac{1}{2}\, m\omega^2(A^2 - x^2) = \frac{1}{2}\, m\omega^2 x^2$ *or* $A^2 = 2x^2$. *Thus* $x = \frac{A}{\sqrt{2}}$ *where A is the amplitude of oscillations.*
7. What is the necessary condition for the simple harmonic motion to occur?
   Ans : The necessary condition is that the acceleration of the body must be proportional to the displacement and it must be directed towards the mean position. Also the other condition is the inertia and the elastic restoring force.

8 A girl is winging in a swing in the sitting position. How is the period of swing affected if she stands up?

Ans : As the girl stands up the effective length of the pendulum decreases as the centre of gravity rises up. As T is directly proportional to square root of length, the time period decreases.

9 A pendulum clock is taken to moon. will it gain or lose time?

Ans : On the moon the acceleration due to gravity is less compared to that on the earth. As T is inversely proportional to the square root of g, the value T increases on moon, i.e. it takes more time for one oscillation. Thus pendulum clock loses time.

10 Why are forced vibration called so?

Ans : A forced vibration corresponds to an external periodic force acting on a body to maintain the oscillations. Thus it is called forced vibrations.

11 What is meant by beat frequency of a coupled oscillator?

Ans : In cases of a coupled oscillator two pendulums are oscillating either in phase of in out of phase. The difference in the frequencies between the out of phase and in phases oscillations is called the beat frequency.

12 Will beat frequency of a coupled oscillator depends upon amplitude of the coupled oscillations.

Ans : No, the beat frequency does not depend on the amplitude of oscillation.

13 What is the condition for minimum time period of a compound pendulum?

Ans : the condition for minimum time period of a compound pendulum is that the variation of time period with length is zero and thus distance between the point of suspension and the centre of gravity is equal to the radius of gyration i.e. $l = k$.

14 What is the condition for maximum time period of a compound pendulum?

Ans : The condition for maximum time period is that the distance of point of suspension to the centre of gravity is zero. i.e. the point of suspension passes through the centre of gravity. This leads to infinite time period.

Elasticity

Syllabus : Hooke's law, Stress – Strain diagram, definitions of three elastic moduli; Relationship between three elastic constants (derivation); Poisson's ratio; Work done in stretching a wire; Bending of beams; Bending moment, Theory of single cantilever, Couple per unit twist, Torsional oscillations

Elasticity is a property of material bodies by virtue of which they regain their original shape and size after the deforming forces are removed.

Stress and strain : When a force acts on a body producing deformation, the internal reactional force which tries to restore the original condition is called **stress**. It is measured in terms of force per unit area. Its unit is Nm^{-2}. The stress may be (i) longitudinal (force acting along one direction such as length), (ii) tangential (force acting along the surface) and (iii) normal (force acting perpendicular to the surface).

When a deforming force acts on a body, the dimensions of the body such as length, shape or volume undergoes a change. The ratio of change in dimension to its original dimension is called **strain**. It has no unit and is a dimensionless quantity.

If a force acting on the body results in change of length, then the strain is longitudinal. The **longitudinal strain** is the ratio of change in length to the original length.

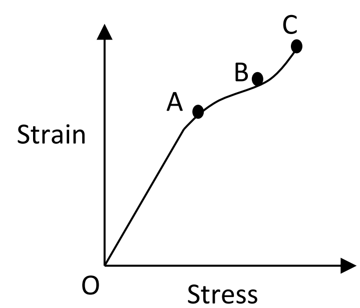
If a tangential force is applied on a body, there is a shearing strain. The body is sheared through an angle θ. This shearing angle is called **shearing strain**.

If a normal force (force acting perpendicular to the surface of the body) is applied on the body which changes its volume, then the ratio of change in volume to the original volume is called **volume strain**.

Hooke's law Within the elastic limit, the stress is directly proportional to strain.

Stress $\propto$ strain $\frac{stress}{strain} = constant = modulus\ of\ elasticity(E)$. E depends on the nature of the material and its unit is Nm^{-2}.

Stress- strain diagram : It is a graph showing the variation of strain with stress (load). Consider a wire stretched by applying load (weight). As the stress is increased, the strain (extension of length) also increases linearly obeying Hooke's law. This is indicated by the straight line OA. The point A indicates the **elastic limit** i.e. within this limit of stress the wire will regain its original length if the load is removed.



When the load is further increased beyond elastic limit, the extension of wire is such that, it will not regain its original length if the load is removed. Here Hooke's law is not obeyed. The variation is non linear as shown by the path A to B. Point B is **yield point.**

If the load is further increased, the extension increases drastically. Also area of cross section of the wire (thickness of the wire) decreases drastically. This is called **necking**. The extension increases until the wire breaks. This point is called the **breaking point** i.e. C. The stress required to break the wire is called the **breaking load**. The ratio of the breaking load to the original area of cross section of the wire is called the **breaking stress** or ultimate strength of the material.

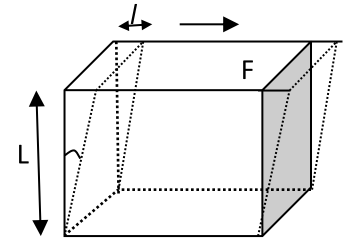
It is observed that even within the elastic limit, the material takes some time to regain its original dimension after the load is removed. This delay is called **elastic after effect**.

In the design of structures, care should be taken to see that material is well within the breaking stress. The fraction of the breaking stress to be maintained is called **working**

stress. The ratio of the ultimate strength to working stress is called the **factor of safety**. For most of the materials this factor is between 5 and 10.

Moduli of elasticity

1. **Young's modulus (q)** – It is the ratio of longitudinal stress to the longitudinal strain within the elastic limits. $q = \frac{longitudinal\ stress}{longitudinal\ strain} = \frac{F/A}{l/L} = \frac{F\,L}{l\,A}$, where F is the force or the load applied to a wire, A is its area of cross section, l is the change in length and L is the original length of a wire.
2. **Rigidity modulus (n)** – It is the ratio of tangential stress to the shearing strain under elastic limits. $n = \frac{tangential\ stress}{shearing\ strain} = \frac{F/A}{\theta} = \frac{F/A}{l/L} = \frac{F\,L}{l\,A}$. (shearing strain = $\theta = tan\theta = \frac{l}{L}$.



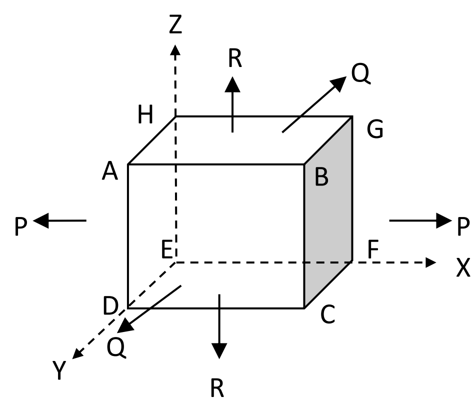
3. **Bulk modulus (k)** – It is the ratio of volume stress or the pressure and the volume strain under elastic limit. $k = \frac{volume\ stress}{volume\ strain} = \frac{F/A}{-\Delta V/V} = \frac{PV}{-\Delta V}$.

Poisson's ratio (σ) - The ratio of lateral strain to the longitudinal strain within the elastic limits is called **poisson's ratio** of the material of the body.

When a force acts on a wire along its length, extension of the wire occurs along the length and contraction of the wire occurs perpendicular to it, thickness or the radius decreases. The ratio of increase in length to original length is the longitudinal strain (α). The ratio of decrease in radius (r) to the original radius (R) is the **lateral strain (β)**.

$$\sigma = \frac{lateral\ strain}{longitudinal\ strain} = \frac{\beta}{\alpha} = \frac{r/R}{l/L} = \frac{rL}{lR} .$$

Relation between elastic constants q, n, k and σ)



Consider a solid cube ABCDEFGH of unit side (all the sides are of equal length equal to 1) as shown.
Let P, Q and R be the forces acting parallel to X, Y and Z axes as shown. The forces P and P acting along X direction, produce elongation along X direction and contraction along Y and Z direction.
As the side of the cube is of unit length, the elongation along X axis is the longitudinal strain α and the contraction along Y and Z axis is the lateral strain β.

To find k

The Young's modulus $q = \frac{longitudinal\ stress}{longitudinal\ strain} = \frac{F/A}{l/L}$ As the cube sides are of unit length, area is equal to one and force is represented by p. Also longitudinal strain = α.

Thus $q = \frac{P}{\alpha}$ or $\boldsymbol{\alpha = \frac{P}{q}}$

The poisson's ratio $\sigma = \frac{\beta}{\alpha}$ or $\beta = \sigma\alpha$ or $\boldsymbol{\beta = \frac{\sigma P}{q}}$

Along X axis, elongation is $\alpha = \frac{P}{q}$ and lateral contraction along Y axis produced by P is $\beta = \frac{\sigma P}{q}$ and lateral contraction along Z axis produced by P is $\beta = \frac{\sigma P}{q}$. Similar is the case with other two directions as shown below.

Elongation	Contraction	Contraction
Along X $\alpha = \frac{P}{q}$	Along Y $\beta = \frac{\sigma P}{q}$	Along Z $\beta = \frac{\sigma P}{q}$
Along Y $\alpha = \frac{Q}{q}$	Along Z $\beta = \frac{\sigma Q}{q}$	Along X $\beta = \frac{\sigma Q}{q}$
Along Z $\alpha = \frac{R}{q}$	Along X $\beta = \frac{\sigma R}{q}$	Along Y $\beta = \frac{\sigma R}{q}$

Thus net elongation along X axis is $\frac{P}{q} - \frac{\sigma Q}{q} - \frac{\sigma R}{q}$.

Similarly, the net elongation along Y axis is $\frac{Q}{q} - \frac{\sigma R}{q} - \frac{\sigma P}{q}$.

and the net elongation along Z axis is $\frac{R}{q} - \frac{\sigma P}{q} - \frac{\sigma Q}{q}$.

If P = Q = R, then the net elongation along X axis is $\frac{P}{q} - \frac{\sigma P}{q} - \frac{\sigma P}{q} = \frac{P}{q}(1 - 2\sigma)$.

Similarly the net elongation along Y axis is $\frac{P}{q}(1 - 2\sigma)$ and along Z axis is $\frac{P}{q}(1 - 2\sigma)$.

Hence the new volume of the cube is $\left[1 + \frac{P}{q}(1 - 2\sigma)\right]^3 = 1 + \frac{3P}{q}(1 - 2\sigma)$ (original volume is 1). Hence the change in volume is $\frac{3P}{q}(1 - 2\sigma)$.

The bulk modulus is $k = \frac{volume\ stress}{volume\ strain} = \frac{P}{\frac{3P}{q}(1-2\sigma)} = \frac{q}{3(1-2\sigma)}$.

Thus $\boldsymbol{k = \frac{q}{3(1-2\sigma)}}$ …….. (1)

To find n

The elongation in one direction and an equal contraction in the perpendicular direction is equivalent to shearing strain θ. If P = - Q and R = 0, then the net elongation along the X axis is $\frac{P}{q} + \frac{\sigma P}{q} = \frac{P}{q}(1 + \sigma)$.

The net contraction along Y axis is same as above i.e. $\frac{P}{q}(1 + \sigma)$

Thus the shearing strain is $\theta = \frac{P}{q}(1+\sigma) + \frac{P}{q}(1+\sigma) = \frac{2P}{q}(1+\sigma)$.

The rigidity modulus of the material is $n = \frac{tangential\ stress}{shearing\ strain} = \frac{P}{\theta} = \frac{P}{\frac{2P}{q}(1+\sigma)}$.

Thus $\boldsymbol{n = \frac{q}{2(1+\sigma)}}$ ....(2)

To find q

From equation (1) $q = 3k(1-2\sigma)$ or $2\sigma = 1 - \frac{q}{3k}$ .....(3)

From equation (2) $q = 2n(1+\sigma)$ or $2(1+\sigma) = \frac{q}{n}$ or $2\sigma = \frac{q}{n} - 2$ ...(4)

Comparing (3) and (4) $1 - \frac{q}{3k} = \frac{q}{n} - 2$ or $\frac{q}{n} + \frac{q}{3k} = 3$ or $\frac{1}{n} + \frac{1}{3k} = \frac{3}{q}$.

or $\boldsymbol{q = \frac{9nk}{3k+n}}$ ....(5)

To find σ

From equations (1) and (2) we have $q = 3k(1-2\sigma)$ $q = 2n(1+\sigma)$

Comparing the above equations, $3k(1-2\sigma) = 2n(1+\sigma)$

or $3k - 6k\sigma = 2n + 2n\sigma$ or $2n\sigma + 6k\sigma = 3k - 2n$

Thus $\boldsymbol{\sigma = \frac{3k-2n}{2n+6k}}$

The limiting values of σ : From equations (1) and (2) we have $3k(1-2\sigma) = 2n(1+\sigma)$. The values of n and k are always positive. For the above equation to be positive, σ should be positive. In RHS of above equation, σ is already positive. For LHS to be positive, $2\sigma < 1$ or $\sigma < \frac{1}{2}$ $or\ \sigma < 0.5$.

If σ were to be negative, LHS is positive but RHS is positive only when σ should not be less than -1. But negative value of σ means the body laterally elongated and longitudinally contracted when stretched longitudinally which is not possible. Thus the limiting value of σ is between 0 and 0.5.

Work done in stretching

When a body is stretched leading to strain, work is said to be done which is stored in the body as potential energy. Consider a wire of length L and area of cross section A being stretched by application of a load F. This results in the increase of the wire by l .

The Young's modulus is $q = \frac{F/A}{l/L} = \frac{FL}{lA}$ or $F = \frac{qAl}{L}$ .... (1)

Work done in stretching the wire by a small length of dl is $dW = F\,dl$

Total work done in producing a stretching from 0 to l is $W = \int_0^l F\,dl = \frac{qA}{L}\int_0^i l\,dl$

Or $W = \frac{qA}{L}\left[\frac{l^2}{2}\right]_0^l = \frac{qA}{L}\frac{l^2}{2}$ or $W = \frac{1}{2}\frac{qAl}{L}\, l = \frac{1}{2} \times F \times l$

Thus $done = \frac{1}{2} \times load \times elongation$. The volume of the wire is $= AL$.

Hence work done per unit volume of the wire is $W' = \frac{1}{2}\frac{F\,l}{AL} = \frac{1}{2} \times \frac{F}{A} \times \frac{l}{L}$.

Thus work done per unit volume is $\boldsymbol{W' = \frac{1}{2} \times stress \times strain}$

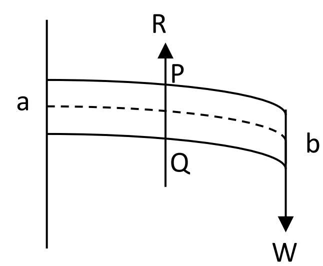
Bending of beams

A beam is defined as a rod or a bar of uniform cross section whose length is very large than its thickness so that the shearing stresses are small and can be neglected.

A beam is said to be made of a number of layers one above the other and each layer is made of number of parallel metallic fibers called longitudinal filaments.

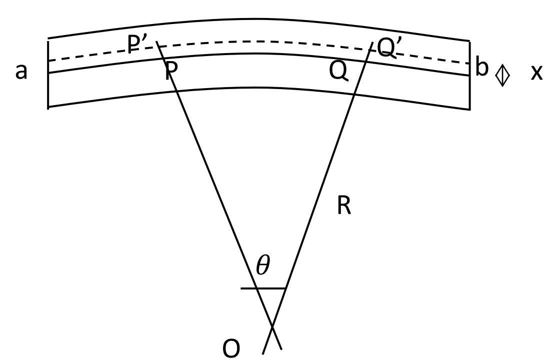
When a beam is bent as shown, the upper layers elongate and lower layers contract. A layer at the centre neither contracts or elongates. This layer is called neutral layer or neutral surface. The line of interaction between the neutral layer and the plane of bending is called neutral axis.

Bending moment : Consider a beam fixed at one end as shown. ab is the neutral axis. When weight W acts downwards at one end, the beam bends. R is the reaction to W acting vertically upwards. These two forces constitute a couple tending to rotate the beam called bending couple. As there is no rotation, an equal and opposite couple acts called restoring couple.



Due to elastic property of the material, a restoring couple acts on beam. At the equilibrium position, the bending couple is equal and opposite to the restoring couple. The moment of the restoring couple is called the bending moment.

Expression for the bending moment



Consider a beam bent as shown. Let R be radius of curvature of the neutral axis ab which subtends an angle θ at the centre of curvature O. PQ is the filament of the beam on the neutral axis. Consider another filament P'Q' at a distance x above PQ. The length of the filament P'Q' is same as that of PQ when it is not loaded or bent. Thus Original length $PQ = R\theta$.

Elongated length due to load = P'Q' = $(R + x)\theta$

Thus the increase in length = P'Q' - PQ = $(R + x)\theta - R\theta = x\theta$

Therefore linear strain = $\frac{increase\ in\ length}{original\ length} = \frac{x\theta}{R\theta} = \frac{x}{R}$

If q is the Young's modulus of the material of the beam, then $q = \frac{linear\ stress}{lineal\ strain}$.

Thus linear stress on P'Q' = $q \times linear\ strain$ = $\frac{qx}{R}$

If f is the internal elastic force which produces the stress, then stress = $\frac{f}{a}$, where a is the area of cross section of the filament.

Then $\frac{f}{a} = \frac{qx}{R}$ or $f = \frac{q\,a\,x}{R}$.

The moment of this force about the neutral axis = $f \times x = \frac{q\,a\,x^2}{R}$

Thus the bending moment is the sum of all the moments of all such internal elastic forces acting over the whole cross section of the beam.

$Bending\ moment = \sum q\ \frac{a\,x^2}{R} = \frac{q}{R}\sum a\,x^2 = \frac{q}{R}\ I$ where $I = \sum a\,x^2$ is called the geometrical moment of inertia of the cross section of the beam about an axis through its centre and perpendicular to the plane of bending.

$\boldsymbol{Bending\ moment} = \frac{\boldsymbol{qI}}{\boldsymbol{R}}$ **where** $\boldsymbol{I = \sum a\,x^2}$

If A is the area of cross section of the beam and K is the radius of gyration about the axis, then $I = AK^2$. Thus $\boldsymbol{Bending\ moment} = \frac{\boldsymbol{qAK^2}}{\boldsymbol{R}}$

Case 1: For a beam of rectangular cross section having breath b and thickness d,

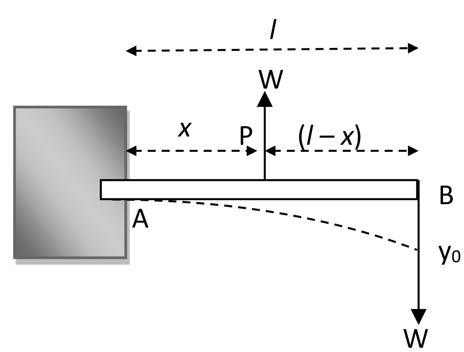
$A = bd$ and $K^2 = \frac{d^2}{12}$ Thus $I = AK^2 = \frac{bd^3}{12}$

Case 2 : For a beam of circular cross section of radius r, $A = \pi r^2$ and $K^2 = \frac{r^2}{4}$.

Thus $I = AK^2 = \frac{\pi r^4}{4}$.

The quantity $qI = qAK^2$ is called the flexural rigidity of the beam and is the bending moment required to bend the beam in the arc of radius unity.

Cantilever - Theory



A cantilever is a beam that is fixed at one end and a load hangs from the other end.

Consider a beam AB of length l that is fixed at A and loaded with weight W at end B.

The beam bends by a distance y_0. Consider a section of the beam at P at distance x from the fixed end. The external bending couple at P is balanced by the internal bending moment as the beam is in equilibrium. This condition is indicated as follows

$W(l-x) = \frac{qI}{R}$ ....(1) where R is the radius of curvature of the neutral axis at P and I is the geometrical moment of inertia of the section of the beam at P.

From differential calculus, R can be represented as $R = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$ ....(2)

If the slope of the beam $\frac{dy}{dx}$ with the horizontal is small, then it can be neglected with respect to 1. The equation (2) becomes $R = \frac{1}{\frac{d^2y}{dx^2}}$ or $\frac{1}{R} = \frac{d^2y}{dx^2}$ .... (3)

Substituting for 1/R from (3) in (1) we get $W(l-x) = qI\frac{d^2y}{dx^2}$

or $\frac{d^2y}{dx^2} = \frac{W}{qI}(l-x)$ ......(4)

integrating (4) we get $\frac{dy}{dx} = \frac{W}{qI}\left(lx - \frac{x^2}{2}\right) + C_1$ ....(5) where C_1 is the constant of integration.

At the end A, x = 0. Thus $\frac{dy}{dx} = 0$, i.e. $C_1 = 0$ (from the above equation)

Hence equation (5) becomes $\frac{dy}{dx} = \frac{W}{qI}\left(lx - \frac{x^2}{2}\right)$

Integrating this equation again, $y = \frac{W}{qI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2$ ......(6)

Again at end A, x = 0. Thus y = 0. From (6) $C_2 = 0$.

Equation (6) becomes $y = \frac{W}{qI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$ ....(7)

If y_0 is the maximum depression of the beam at end B, i.e. at distance $x = l$

Equation (7) can be written as $y_0 = \frac{W}{qI}\left(\frac{ll^2}{2} - \frac{l^3}{6}\right) = \frac{W}{qI}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{W}{qI}\frac{l^3}{3} = \frac{Wl^3}{3qI}$

or $\boldsymbol{y_0 = \frac{Wl^3}{3qI}}$ ....(8) This is the expression for depression of cantilever.

Case 1 : For a beam of rectangular cross section with b as breadth and d as thickness, $I = \frac{bd^3}{12}$. Thus equation (8) becomes $y_0 = \frac{Wl^3}{3q}\frac{12}{bd^3} = \frac{4Wl^3}{bd^3q}$

Thus the Young's modulus of the material of the beam is $q = \frac{4Wl^3}{bd^3\,y_0}$ ....(9)

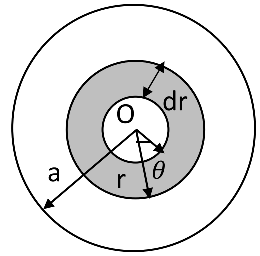
where W = mg.

Case 2 : For a beam of circular cross section $I = \frac{\pi r^4}{4}$ where r is the radius of the beam.

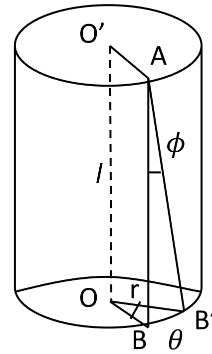
Thus $y_0 = \frac{Wl^3}{3q}\frac{4}{\pi r^4} = \frac{4Wl^3}{3\pi r^4 q}$ or $q = \frac{4Wl^3}{3\pi r^4\,y_0}$ where W = mg.

Twisting of a cylinder - Expression for couple per unit twist

Consider a cylindrical rod of length l and radius a. Let the rod be fixed at the upper end and twisted at the lower end through an angle θ. This results in a line AB on the rod twisted through ϕ to the position AB'.



Cross sectional view



From the diagram, the angle of shear = $\angle BAB' = \phi$.

Also from the triangle BOB', $\angle BOB' = \theta$ and OB' = r.

Thus BB' = $r\theta$ Also $BB' = l\phi$ from triangle $\angle BAB'$.

Comparing the two, $r\theta = l\phi$ or $\phi = \frac{r\theta}{l}$

$Rigidity\ modulus\ n = \frac{shearing\ stress}{angle\ of\ shear}$ or $Shearing\ stress = n \times angle\ of\ shear$

Thus $shearing\ stress = n\,\phi = n\,\frac{r\theta}{l}$

$Shearing\ stress = \frac{shearing\ force}{area}$ or $shearing\ force = stress\ \times area\ on\ which\ force\ acts$

Thus $shearing\ force = n\,\frac{r\theta}{l} \times 2\pi r\,dr$ where area = $2\pi r\,dr$

$$shearing\ force = \frac{2\pi n\theta r^2}{l} dr$$

Moment of this force about the axis OO' = $\frac{2\pi n\theta r^2}{l} dr \times r = \frac{2\pi n\theta r^3}{l}\,dr$

The twisting couple over the entire cylinder = $C = \int_0^a \frac{2\pi n\theta r^3}{l}\,dr$

Hence the couple per unit twist, $c = \frac{C}{\theta} = \int_0^a \frac{2\pi n r^3}{l}\,dr = \frac{2\pi n}{l} \int_0^a r^3 dr$

Couple per unit twist $c = \frac{2\pi n}{l}\left[\frac{r^4}{4}\right]_0^a$ or $\mathbf{c} = \boldsymbol{\frac{\pi n a^4}{2l}}$

Couple per unit twist is also called the torsional rigidity. It is the restoring couple per unit twist due to torsional reaction.

Torsional oscillations : Consider a body like disc, bar or cylinder suspended by a wire fixed to a chuck nut at the top. When the wire is given a small twist and let go, the body oscillates in the horizontal plane with the wire as the axis. These oscillations are referred to as torsional oscillations.

Expression for time period of torsional oscillations

If I is the moment of inertia of the body and α is the angular acceleration of the system, then, the external deflecting couple = $I\alpha$

This results in the internal restoring couple = $c\theta$ where c is the couple per unit twist of the suspension wire and θ is the angular displacement.
For equilibrium to exist the above two couples must be equal and opposite.

i.e. $I\alpha = -\,c\theta$ As $\alpha = \frac{d^2\theta}{dt^2}$, the above equation becomes $I\frac{d^2\theta}{dt^2} = -\,c\theta$

Or $I\frac{d^2\theta}{dt^2} + c\theta = 0$ or $\frac{d^2\theta}{dt^2} + \left(\frac{c}{I}\right)\theta = 0$ .....(1)

This equation represents equation of simple harmonic motion with $\left(\frac{c}{I}\right) = \omega^2$ ....(2)

ω is angular frequency. In terms of time period $\omega = \frac{2\pi}{T}$ or $\omega^2 = \frac{4\pi^2}{T^2}$ ...(3)

Comparing (2) and (3) we get $\frac{4\pi^2}{T^2} = \frac{c}{I}$ or $T^2 = 4\pi^2\frac{I}{c}$

Thus the time period of torsional oscillations is given by $\boldsymbol{T = 2\pi\sqrt{\frac{I}{c}}}$

Note : In general, elastic moduli decreases with increase in temperature and for small changes, the variation is approximately linear. The Young's modulus varies with temperature according to the equation $q = q_0\, e^{-bT}$. The rigidity modulus and bulk modulus also decreases with temperature. Also the elastic constants of some substances increases with pressure and others decreases with pressure.

PART A : Descriptive questions (8 marks each)

1. (a) State Hooke's law. Draw stress strain diagram.
   (b) Deduce the relation between the three moduli of elasticity.
2. (a) What is poisson's ratio? Write a note on limiting values of poisson's ratio.
   (b) Derive an expression for the work done in stretching a wire.
3. What is bending moment? Derive an expression for the bending moment of a beam.
4. (a) What is flexural rigidity? Derive an expression for the couple per unit twist of a material of a wire.
   (b) Write a note on torsional oscillations.
5 What is a cantilever? Obtain an expression for the depression at the free end of a thin light beam clamped horizontally at one end and loaded at the other.
6 (a) Define (i) neutral axis (ii) neutral surface of an elastic beam.
   (b) Derive an expression for the depression of a cantilever.
7 (a) What is strain? Mention the different types of strain.
   (b) Derive an expression for couple per unit twist of the material of the wire.
8 Write the note on torsional oscillations. Obtain an expression for time period of the oscillation.

PART-B : Numerical problems (4 marks each)

1 What force is required to stretch a steel wire $1\ cm^2$ in cross section to double its length? Young's modulus of steel is q = $20\times10^{10} N/m^2$.
[Hint : $q = \frac{F\,L}{l\,A}$ $or\ F = \frac{q\,l\,A}{L}$ $where\ A = 1cm^2 = 1\times10^{-4}m^2\,, l = 2L\,,\ F = 40\times10^6\ N$]

2 Find the work done in stretching a wire of cross-section $2\ mm^2$ (sq. mm) and length $4\ m$ through $0.2\ mm$. Young's modulus of the material of the wire is $20\times10^{10}\ Nm^{-2}$.

[Hint : $W = \frac{1}{2} F\, l,\ As\quad F = \frac{q\, l\, A}{L}\quad W = \frac{1}{2}\ \frac{q\, l^2\, A}{L} = 2\times10^{-3}\, J$]

3 Calculate the longest length of steel wire that can hang vertically without breaking. Breaking stress for steel is 8×10^{8} Nm^{-2}, density of steel is 8000 kgm^{-3}.

[Hint : $Stress = \frac{F}{A} = \frac{mg}{A} = \frac{V\rho\, g}{A} = \frac{LA\,\rho g}{A} = L\rho g,\ Thus\ L = \frac{stress}{\rho\, g} = 1.02\ \times10^{4}\ m$]

4 Find the energy stored in a wire 5 m long and 0.001 m in diameter when it is stretched through 0.003 m by a load. Given : Young's modulus of the material is 20×10^{10} Nm^{-2}.

[Hint : $W = \frac{1}{2}\ \frac{q\, l^2\, A}{L}\quad where\ A = \pi r^2\ and\ r = \frac{d}{2} = 5\ \times10^{-4},\ W = 565.2\, J$]

5 A load of 2 kg produces an extension of 1 mm in a wire of 2 m length and 1 mm in diameter. Find the Young's modulus of the material of the wire.

[Hint $q = \frac{F\, L}{l\, A}$, $where\ F = mg\ \ and\ A = \pi r^2,\quad q = 7.48\ \times10^{10}\ Nm^{-2}$]

6 Calculate the that must be suspended from a steel wire of 1 mm in diameter to produce an elongation of 0.02% of its original length. Q = $20\times10^{10} Nm^{-2}$.

[Hint : $q = \frac{F\, L}{l\, A}$, $F = mg,\ A = \pi r^2,\ l = \frac{0.02}{100} L,\ m = 3.204\ kg$]

7 The end of a cantilever is depressed by 0.001 m under certain load. Calculate the depression for the same load for another cantilever of the same material two times in length, two times in breadth and three times in thickness. [Hint : $y_0 = \frac{Wl^3}{3qI},\quad where\ I = \frac{bd^3}{12},\ l_2 = 2l_1,\ b_2 = 2b_1,\ d_2 = 3d_1,\ To\ find\ \frac{y_{02}}{y_{01}}\ and\ hence\ y_{02} =\ 7.4 \times\times\times 10^{-5} m$]

8 A steel wire of radius 1 mm is bent into an arc of a circle of radius 50 cm. Calculate the bending moment. Given : Young's modulus is 20×10^{10} Nm^{-2}.

[Hint : $Bending\ moment = \frac{q\, I}{R}\ \ where\ I = \frac{\pi r^4}{4},\ R = 50\ cm,\ \ r = 1\ mm, BM = 0.314\ SI\ units$]

9 A cube of aluminium of side $10\ cm$ is subjected to a shearing force of $100\ N$. The top surface of the cube is displaced by $0.01\ cm$ with respect to the bottom. Find the rigidity modulus of the aluminium. [Hint : $n = \frac{F}{A\theta}\quad where\ \ \theta = \frac{l}{L} = \frac{0.01}{10},\ \ A = L^2,\ n = 1\times10^{7} Nm^{-2}$]

10 A steel wire of diameter 3.8×10^{-4} m and length 4.2 m extends by .00182 m under a load of 1 kg and twists by 1.2 radians when subjected to a total torsional torque of 4×10^{-5} Nm at one end. Find the values of young's modulus, rigidity modulus and the poisson's ratio.

[Hint : $q = \frac{F\, L}{l\, A}$, $F = mg,\ A = \pi r^2,\ q = 2\times10^{11} Nm^{-2},\ C = \frac{\pi\, n\, r^4}{2L}\theta,\ n = 6.64\ \times10^{10} Nm^{-2}, n = \frac{q}{2(1+\sigma)}$, $\sigma = 0.457$]

11 A metal cube of side 1m is subjected to a uniform force acting normally on the whole surface of the cube. If the volume changes by $1.5\text{X}10^{-5}$ m^3 and if the pressure is 10^6 Pa find the bulk modulus of the metal. [Hint : $k = \frac{PV}{\Delta V} = 0.66\ \times10^{11} Nm^{-2}$]

12 A wire of length 1 mand diameter $10^{-3} m$ is fixed at one end and twisted at the other end through an angle of 70^0 by applying a couple of 0.01 Nm. Calculate the rigidity modulus of the material of the wire. [Hint : $n = \frac{2\, l\, C}{\pi r^4 \theta} = 8.34\ \times10^{10}\ Nm^{-2}\quad where\ \ \theta = 70\times\frac{\pi}{180}\ rad$]

13 A circular disc of mass 0.8 kg and radius 0.1 m is suspended through its centre and perpendicular to plane, by using wire of length 1 m and radius 0.5 mm. If the period of torsional oscillations is 1.23 seconds, calculate the rigidity modulus of material of the wire.

[Hint : $T = 2\pi\sqrt{\frac{I}{c}}$, $I = \frac{MR^2}{2}$, $find\ couple\ per\ unit\ twist = c = 0.104\ then\ use\ n = \frac{2\,l\,c}{\pi r^4} = 1.06 \times 10^{12}\ Nm^{-2}$]

14 A uniform rod of length 1 m is clamped horizontally at one end. A weight of 0.1 kg is attached at the free end. Calculate the depression at the mid pointof the rod. The diameter of the rod is 0.02 m. q = $1\times 10^{01} Nm^{-2}$.

[Hint : $y = \frac{W}{qI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$, $W = mg,\ I = \frac{\pi r^4}{4}$, $l = 1\,m\ and\ x = 0.5\,m, y = 1.3 \times 10^{-3} m$]

15 One end of a steel wire of length 0.25 m and radius 0.002 m is fixed. If the rigidity modulus n = $8 \times 10^{10}\ Nm^{-2}$, find the work done in twisting the free end of the wire through 45^0.

[Hint :$W = \frac{1}{2}\ c\theta^2$,$where\ c = \frac{C}{\theta}$ $and\ c = \frac{\pi n r^4}{2\,l}$, $W = 2.477\ J$]

16 A body suspended symmetrically from the lower end of a wire of 1 m long and 0.00122 m in diameter, oscillates about the wire as axis with the period of 1.25 s. If n = $8 \times 10^{10}\ Nm^{-2}$, find the moment of inertia about the axis of rotation.

[Hint : $T = 2\pi\sqrt{\frac{I}{c}}$, $c = \frac{\pi n r^4}{2\,l}$, $I = 6.88 \times 10^{-4}\ kgm^2$]

PART C : <u>Conceptual questions (2 marks each)</u>

1. What is the torque acting on a body when the force is applied in the direction of the radius vector? Justify your answer.
   Ans : Zero, As the torque is the cross product of force and the perpendicular distance between the point of force application and the axis, $\tau = r \times F = rF\sin\theta$. *If the force is along the radius vector* $\theta = 0$. *Thus torque is also zero.*
2. Poisson's ratio of any material cannot be less than -1. Explain.
   Ans : Poisson's ratio σ should not be less than -1. Negative value of σ means the body laterally elongated and longitudinally contracted when stretched longitudinally which is not possible. Thus the limiting value of σ is between 0 and 0.5.
3. Young's modulus of a material cannot be greater than 3 times its rigidity modulus. Explain.
   Ans : If the Young's modulus of a material is three times its rigidity modulus, then the volume elasticity will be infinity which is not possible. Also the poisson's ratio will be greater than 0.5 which is also not possible if q is greater than 3n.
4. Explain why steel girders and rails are made in the form if I section.
   Ans : By having a cross section as I, the steel girder is not bent appreciably as its material has a large value of q and it is short and its breadth shorter than depth. This provides a high bending moment and also lot of material is saved.
5. A hollow rod is a better shaft than a solid one of the same material, mass and length. Explain.
   Ans : Average shear stress in a hollow shaft will be higher compared to solid shaft and its value is more closer to the maximum shear stress. Hollow shaft has greater strength to weight ratio.
6. Steel is more elastic than rubber. Explain.
   Ans : Steel comes back to its original shape faster than rubber when the deforming forces are

removed. For a given stress, strain is much smaller in steel than in rubber.

7. In the case of bending of a rod Young's modulus only comes into play and not rigidity modulus, even though there is a change in the shape. Explain.
   Ans : it is a case of nearly pure bending where bending moment is applied to a beam without simultaneous presence of axial, shear or torsional forces. Transverse sections of a rod which are plane before bending will remain plane even after bending.
8. In automobiles, spring made of steel is preferred over copper. Explain.
   Ans : A large restoring force is developed on being deformed in a good spring. This in turn depends on elasticity of the material. As the Young's modulus of steel is higher than copper, steel springs are preferred.
9. What happens to the extension and maximum load a wire can bear if it is cut into half?
   Ans : The extension will be reduced to half. But there is no effect on the maximum load it can bear because the maximum load = breaking stress × area. Here area is a constant.
10. Why are the metal bridges declared unsafe after long use?
   Ans : If the bridges are used for long time, due to alternate cycles of stress and strain, the bridge loses its elastic property and finally reaches a condition called elastic fatigue. At this stage for a given stress, the strain produced is very large and hence the bridge may collapse.