Chapter 1
Linear Algebra

In this chapter, we study another algebric structure called the vector space , the basis and dimension of a vector space, linear transformation and Eigen values and Eigen vectors of a linear transformation .

1.01 : vector spaces

Definition : Let F be a field and V be a non- empty set. In V, we define the operations of addition and scalar multiplication $\alpha+\beta$ and c α where $\alpha,\beta\in$ Vand c ∈ F. Then the set V is called a *vector space* over the field F if the following axioms are satisfield:

(V_1) (V,+) is an Abelian group.

(V_2) (i) c. $(\alpha+\beta)=c.\alpha+c.\beta$

(ii) $(c_1+c_2)\alpha=c_1\alpha+c_2\alpha$ } (Distributive axioms)

(V_3) $(c_1c_2)\alpha=(c_1)(c_2\alpha)$

(V_4) $1.\ \alpha=\alpha$

$\forall\alpha,\beta\in V$ and $c,c_1,c_2,\in F$

The vector space over the field F is denoted by V (F). The elements of F are called *scalars* and the elements of V are called *vectors*.

The identity of the group (V, +) is denoted by 0 and is called the zero vector or null vector which is unique.

Worked Examples :

(1) The set of all ordered pairs (x_1, x_2) of the elements of the field of real numbers forms a vector space w.r.t.

addition and scalar multiplication defined as
$(x_1,x_2)+(y_1,y_2)=(x_1+y_1, x_2+y_2)$
$c.(x_1,x_2)=(cx_1,cx_2)$

Solution : Let $V(R)=\{(x_1,x_2)|\ x_1,x_2\in R\}$.

Let $\alpha,\beta,\gamma\in V(R)$

$\therefore\ \alpha=(x_1,x_2),\ \beta=(y_1,y_2),\ \gamma=(z_1,z_2)$

Let $c, c_1,c_2,\in F$.

(V_1) (V,+) is an Abelian group

(i) $\alpha+\beta = (x_1,x_2)+(y_1+y_2)$

$= (x_1+y_1,x_2+y_2)$

(ii) $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$

LHS $=\alpha+(\beta+\gamma)$

$=(x_1,x_2)+(y_1+z_1,y_2+z_2)$

$=(x_1+y_1+z_1, x_2+y_2+z_2)$

RHS $=(\alpha+\beta)+\gamma$

$=(x_1+y_1, x_2+y_2)+(z_1,z_2)$

$=(x_1+y_1+z_1,x_2+y_2+z_2)$

$\therefore$ LHS = RHS .

(iii) There exists 0 = (0, 0) ∈ V such that

$(0,0)+(x_1,x_2)=(x_1,x_2)+(0,0)=(x_1,x_2)\ \forall(x_1,x_2)\in V$

(iv) $\forall\ \alpha=(x_1,x_2)\in V$ There exists $-\alpha=(-x_1,-x_2)\in V$ such that

$(x_1,x_2)+(-x_1,-x_2)=(-x_1,-x_2)+(x_1,x_2)=(0,0)=0$

(V) $\alpha+\beta=(x_1,x_2)+(y_1,y_2)$

$=(x_1+y_1, x_2+y_2)$

$=(y_1+x_1, y_2+x_2)$

$=(y_1,y_2)+(x_1,x_2)$

$=\beta+\alpha$

$\therefore$ (V,+) is an ablian group .

(V_2) (i) $c(\alpha+\beta) = c(x_1+y_1,x_2+y_2)$

$=(c(x_1+y_1),c(x_2+y_2))$

$= (cx_1+cy_1, cx_2+cy_2)$
$= (c\,x_1, cx_2) + (cy_1, cy_2)$
$= c\,(x_1, x_2) + c\,(y_1, y_2)$
$= c\,\alpha + c\,\beta.$

(ii) $(c_1+c_2)\,\alpha = (c_1+c_2)\,(x_1, x_2)$
$= (c_1+c_2)\,x_1, (c_1+c_2)\,x_2)$
$= (c_1x_1+c_2x_1, c_1x_2+c_2x_2)$
$= (c_1x_1, c_1x_2) + (c_2x_1, c_2x_2)$
$= c_1(x_1, x_2) + c_2(x_1, x_2)$
$= c_1\alpha + c_2\alpha.$

(V_3) $(c_1\,c_2)\,\alpha = (c_1\,c_2)\,(x_1, x_2)$
$= (c_1\,c_2)\,x_1, (c_1\,c_2)\,x_2$
$= (c_1\,(c_2\,x_1), c_1(c_2\,x_2))$
$= c_1(\,c_2\,x_1, c_2\,x_2))$
$= c_1(\,c_2\,(x_1, x_2)) = c_1(\,c_2\alpha\,)$

(V_4) $1\,.\,\alpha = 1.(x_1, x_2) = (\,1.x_1, 1.x_2)$
$= (x_1, x_2) = \alpha\,.$

$\therefore$ all the axioms of vector space are satisfied .

$\therefore$ V is a vector space over the field of real numbers .

(2) The set of all ordered triplets (x_1, x_2, x_3) over the field of real numbers forms a vector space w.r.t addition and scalar multiplication defined in the same way as in the previous example.

Proof is similar to the proof of the previous problem.

(3) The set all ordered n tuples of the elements of the field F froms a vector space w.r.t addition and scalar multiplication defined as

(i) $(x_1, x_2 \ldots\ldots\ldots\ldots x_n) + (y_1, y_2, \ldots\ldots\ldots\ldots y_n)$
$= (x_1+y_1, x_2+y_2, \ldots\ldots\ldots\ldots\ldots x_n+y_n)$

(ii) $c\,(x_1, x_2 \ldots\ldots\ldots\ldots x_n) = (cx_1, cx_2 \ldots\ldots\ldots\ldots, cx_n)$

Proof is as in the previous examples .

The vector space of ordered n triples over the field of real numbers is denoted by V_n (R) or R^n which is called the n dimenstional space.

In particular, if n =2, the vector space is v_2(R) which is the two dimensional plane and if n = 3, the vector space is v_3(R) which is the three dimensional space.

(4) Prove that the set of all real valued continuous (differentiable, integrable) fuctions of X defined in the interval [0,1] is a vector space.

Solution **:** Let V be the set of all real valued continuous functions of x defined in [0,1] .

Let $f,g \in$ v and $c \in$ R. then

$(f+g)\,(x) = f(x) + g(x)$

and $cf\,(x) = (cf)(x)$

(V_1) (V,+) is an abelian group .

(i) If f and g are continuous fuctions, then we know that their sum f + g is also continuous.

(ii) If $f, g, h \in$ V then

$f + (g+h) = (f+g)+h$

Now $[f + (g+h)]\,(x) = f\,(x) + (g + h)(x)$
$= f(x) + [g(x) + h(x)]$
$= [\,f\,(x) + g(x)] + h(x)$
$= (f + g)(x) + h(x)$
$= [\,(f + g) + h\,]\,(x)$

$\therefore$ $f + (g + h) = (f + g) + h.$

(iii) The function 0(x) = 0 is the identity.

$\because$ $(0 + f)\,(x) = 0(x) + f(x) = 0 + f(x) = f(x)$

and $(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$

$\therefore$ $0 + f = f + 0 = f.$

(iv) $(-f)\,x = -[f(x)]$ is the additive inverse of f.

$\because$ $(-f)\,x + f\,(x) = (-f + f)\,(x) = 0\,(x) = 0$

and $f(x) + (-f)\,(x) = [f + (-f)]\,(x) = 0\,(x) = 0$

(v) (f +g) (x) = f (x) +g(x)
= g (x) +f (x)
= (g +f) (x)
$\therefore$ f +g = g + f
$\therefore$ (V, +)is an abelian group.

(v_2) (i) c (f +g) = cf + cg.
[c(f +g)](x) = c[(f+g)x]
= c[f(x)+g(x)]
= cf(x) +cg(x)
= (cf) (x) + (cg) (x)
= (cf +cg) (x)
(ii) $(c_1+c_2)f = c_1f+c_2f$
$(c_1+c_2)f\ (x) = c_1f(x) + c_2f(x)$
$= c_1f+c_2f(x)$
$\therefore$ $(c_1+c_2)f = c_1f+c_2f$

(v_3) $(c_1 c_2) f = c_1 (c_2f)$
$(c_1c_2) f(x) = [(c_1c_2) f] (x)$
$= [c_1(c_2f)] (x)$
$= c_1 (c_2f)(x)$
$\therefore$ $(c_1c_2) f = c_1 (c_2f)$
(v_4) 1. f = f
(1.f) (x) = 1.f (x) =f(x)
$\therefore$ 1. f = f
$\therefore$ all the axioms of vector space are satisfied
$\therefore$ v forms a vector space .

(5) The set of all convergent sequences of real number is a vector space over the field of real numbers.

***Solution*:** Let $\alpha = \{\alpha_1, \alpha_{2,} ....\alpha_n ....\}$

$\beta = \{\beta_1, \beta_2, ...\beta_n ......\}$

$\gamma = \{\gamma_1, \gamma_2, ...\gamma_n .....\}$

be convergent sequences of real numbers.
(V_1) (V,+) is an abelian group.

(i) $\alpha + \beta = \{\alpha_1 + \beta_{1,} \alpha_2 + \beta_2 ...., \alpha_n + \beta_{n,......}\}$
is also a convergent sequence.

(ii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$\alpha + (\beta + \gamma) = \alpha + \{\beta_1 + \gamma_1, \beta_2 + \gamma_2, ...\beta_n + \gamma_{n,..........}\}$

$= \{\alpha_1 + (\beta_1 + \gamma_1), \alpha_2 + (\beta_2 + \gamma_2), ...\alpha_n + (\beta_n + \gamma_n) ....\}$

$= \{(\alpha_1 + \beta_1) + \gamma_1, (\alpha_2 + \beta_2) + \gamma_{2,} .....(\alpha_n + \beta_n) + \gamma_n ....\}$

$= \{\alpha_1 + \beta_1, \alpha_2 + \beta_2 ....\alpha_n + \beta_{n,} ......\} + \{\gamma_1, \gamma_2, ...\gamma_n ....\}$

$= (\alpha + \beta) + \gamma$

(iii) The identity element is { 0} = { 0, 0, ………0,……..}

(iv) If $\alpha = \{\alpha_1, \alpha_2, ...\alpha_n ....\}$ *then*

$-\alpha = \{-\alpha_1, -\alpha_2 ....-\alpha_{n,} ......\}$

is the additive inverse

(v) $\alpha + \beta = \{ \alpha_1+\beta_1, \alpha_2+\beta_{2,......} \alpha_n+\beta_n ....\}$
$=\{ \beta_1+\alpha_{1,} \beta_2+\alpha_2 .... \beta_n+ \alpha_n ......\}$
$= \beta + \alpha$

(v_2) $c(\alpha + \beta)$
$=\{c(\alpha_1+ \beta_1), c(\alpha_2 + \beta_{2,}) ....c (\alpha_n + \beta_n) ......\}$
$=\{c\alpha_1+c\beta_1, c\alpha_2 + c\beta_{2,..............} c\alpha_n + c\beta_n ....\}$
$=c\{ \alpha_1, \alpha_2 .... \alpha_{n,........}\}+c \{ \beta_1, \beta_{2,......} \beta_{n............}\}$
$= c\alpha + c\beta$
$(c_1+ c_2) \alpha = (c_1+ c_2) \{ \alpha_1, \alpha_2 .... \alpha_{n,........}\}$
$= \{(c_1+ c_2)\alpha_1, (c_1+ c_2)\alpha_2 ....(c_1+ c_2) \alpha_{n,........}\}$
$= \{c_1\alpha_1+c_2\alpha_{1,} c_1\alpha_2+ c_2\alpha_2 ......c_1\alpha_n+ c_2\alpha_n ....\}$
$=\{c_1\alpha_{1,}c_1\alpha_2 ..... c_1\alpha_n\}+ \{c_2 \alpha_1, c_2\alpha_{2,.....} c_2\alpha_n ....\}$
$= c_1\{ \alpha_{1,,} \alpha_2 ..... \alpha_n\}+ c_2\{ \alpha_1, \alpha_{2,....} \alpha_n ....\}$
$= c_1\alpha + c_2\alpha$.

(v_3) $(c_1 c_2)\alpha = (c_1 c_2) \{ \alpha_{1,} \alpha_2 ..... \alpha_n ....\}$

$= \{(c_1 c_2)\alpha_1, (c_1 c_2)\alpha_2 \ldots.. (c_1 c_2)\ \alpha_n \ldots\ldots\}$
$= \{\ c_1 (c_2\alpha_1)\ , c_1 (c_2\alpha_2), \ldots\ldots.\ c_1 (c_2\ \alpha_n) \ldots\ldots\}$
$= c_1\{c_2\alpha_1\ , c_2\alpha_2, \ldots\ldots.c_2\ \alpha_n \ldots\ldots\}$
$=\ \ c_1c_2\alpha_1\ ,\ \alpha_2, \ldots\ldots.\ \alpha_n \ldots\ldots\}]$
$=\ c_1(c_2\ \alpha)$

$(v_4) 1.\alpha = 1\{\alpha_1, \alpha_2 \ldots.. \alpha_n \ldots\ldots.\} = \{1.\ \alpha_1,\ 1.\alpha_2 \ldots.1.\alpha_n \ldots.\}$
$= \{\ \ \alpha_{1,},\ \alpha_2 \ldots..\ \alpha_n \ldots\ldots.\}$
$=\ \ \alpha\ .$

$\therefore$ The set of all convergent sequences forms sector space over the field of real numbers

(6) The set of all ordered n – triples of complex numbers forms a vector space over the field of complex numbers. This is denoted by C^n

Solution : Let V = $\{(z_1, z_2, \ldots\ldots\ldots z_n) \mid z_1, z_2, \ldots\ldots\ldots z_n \in c\}$

Let $\alpha,\ \beta, \gamma \in V$

$\therefore\ \alpha\ = (\alpha_1, \alpha_2 \ldots..\ \alpha_n)\ ,\ \beta = (\ \ \beta_1, \beta_2 \ldots\ldots\ldots \beta_n)$

$\gamma\ = (\gamma_1, \gamma_2 \ldots\ldots\ldots.. \gamma_n)$

$\alpha + \beta\ = (\alpha_1 + \beta_1,\ \alpha_2 + \beta_2 \ldots\ldots\ldots\ldots\ \alpha_n + \beta_n)$

$c\alpha\ \ =\ \ (c\alpha_1, c\alpha_2 \ldots..\ c\alpha_n)$. Where $c \in c$.

(V_1) (V,+) is an Abelian group .

(i) $\alpha + \beta\ = (\ \alpha_1 + \beta_1,\ \alpha_2 + \beta_2 \ldots\ldots\ldots\ldots.\alpha_n + \beta_n) \in V$

(ii) $\alpha + (\beta + \gamma)\ = (\alpha +\ \beta) + \gamma$ which can be easily verified

(iii) 0 = (0, 0,0,……….0) is the additive identity

(vi) $\forall\alpha\ = (\alpha_{1,},\ \alpha_2 \ldots..\ \alpha_n) \in V$, there exists

$-\alpha\ = (-\alpha_1, -\ \alpha_2 \ldots..\ -\alpha_n) \in V$ such that

$\alpha + (\ -\alpha\ ) = -\alpha + \alpha = (0,0,\ldots\ldots\ldots\ldots0) = 0$.

(V) $\alpha + \beta\ =\ \alpha + \alpha$

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(V_2) (i) $c\ (\alpha + \beta)\ \ = c\ (\alpha_1 + \beta_1,\ \ \ \ \alpha_2 + \beta_2 \ldots\ldots\alpha_n + \beta_n)$
$= (c(\alpha_1 + \beta_1), c(\alpha_2 + \beta_2) \ldots\ldots\ (\alpha_n + \beta_n))$

$= (c\alpha_1 + c\ \beta_1, c\alpha_2 + c\beta_2,\ \ldots\ldots\alpha_n + c\ \beta_n)$
$= (c\alpha, c\alpha_2 \ldots.\ \ \alpha_n) + (\ c\beta_1, c\beta_2 \ldots\ c\beta_n)$
$= c\ (\alpha_1,\ \alpha_2 \ldots..\ \alpha_n) + (\ \ \beta_1, \beta_2 \ldots\ldots\beta_n)$
$= c\alpha + c\beta$.

(ii) $(c_1 + c_2)\ \ \alpha = c_1\alpha + c_2\alpha$. Which can be verified easily

(V_3) (c_1,c_2) $\alpha = c_1\ (\ c_2\alpha\ )$ which can be verified easily

(V_4) 1. $\alpha\ = 1\ (\alpha_1, \alpha_2, \ldots\ldots\ \ \alpha_n) = (1\alpha_1, 1\alpha_2, \ldots\ldots\ldots.1\alpha_n)$
$=\ (\alpha_1, \alpha_2, \ldots\ldots\ldots.\alpha_n)$

$\therefore$ V is a vector space over the field of complex numbers.

(7) Show that the set V = $\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in R\right\}$ is a vector space over the field of reals R. under usual + and x.

Solution : $A = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}, B = \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}, C = \begin{pmatrix} x_3 & 0 \\ 0 & y_3 \end{pmatrix} \in V$

and $\qquad c_1, c_2, c_3 \in R$

$(v_1)(v, +)$ is an abelian group

(i) $A + B = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 & 0 \\ 0 & y_1 + y_2 \end{pmatrix} \in V$

(ii) $A + (B + C) = (A + B) + C$

$$LHS = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \left(\begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} + \begin{pmatrix} x_3 & 0 \\ 0 & y_3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} x_1 + x_2 + x_3 & 0 \\ 0 & y_1 + y_2 + y_3 \end{pmatrix}$$

$$RHS = \left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}\right) + \begin{pmatrix} x_3 & 0 \\ 0 & y_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_2 + x_3 & 0 \\ 0 & y_1 + y_2 + y_3 \end{pmatrix}$$

$\therefore$ Associative law is valid

(iii) Matrix $0 \in$ V such that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

(iv) – A + A = A + (-A) = 0

(v) $A+B = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 & 0 \\ 0 & y_1+y_2 \end{pmatrix}$

$$= \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} + \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} = B+A$$

Hence (v +) is an abelian group.

(v) (i) $c_1(A+B) = c_1\left(\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}\right)$

$$= c_1\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} + c_1\begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}$$

$$= (c_1+c_2)A = (c_1+c_2)\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}$$

$$= \begin{pmatrix} (c_1+c_2)x_1 & 0 \\ 0 & (c_1+c_2)y_1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1x_1 & 0 \\ 0 & c_1y_1 \end{pmatrix} + \begin{pmatrix} c_2x_1 & 0 \\ 0 & c_2y_1 \end{pmatrix} = c_1A + c_2B$$

$\therefore$ Distributive axioms are valid.

(v_3) $(c_1c_2)A = (c_1c_2)\left[\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}\right]$

$$= \begin{pmatrix} c_1c_2x_1 & 0 \\ 0 & c_1c_2y_1 \end{pmatrix} = c_1\begin{pmatrix} c_2x_1 & 0 \\ 0 & c_2y_1 \end{pmatrix} = c_1(c_2A)$$

(v_4) I is the identity of V w.r.t x because

$$I.A = A.I = A \qquad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $I.c_1 = c_1.I = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} \in V$

All the axioms of vector space are satisfied.

$\therefore$ V is a vector space over the field of real numbers

(8) Show that R^2(R) is not a vector space when + and x are defined as $(\alpha_1,\alpha_2)+(\beta_1,\beta_2) = (\alpha_1+\beta_1, \alpha_2+\beta_2)$

and $\alpha.(\alpha_1,\alpha_2) = (\alpha\alpha_1, \alpha\alpha_2) \forall \alpha_1,\alpha_2,\beta,\beta_2 \in R$

Solution : Let $\alpha = 1$, $\beta = 2$ and $(\alpha_1,\alpha_2) = (3,4)$

$$(\alpha+\beta).(\alpha_1,\alpha_2) = (1+2)(3,4) = (9,4)(9,12)$$

and $\alpha.(\alpha_1,\alpha_2) + \beta.(\alpha_1,\alpha_2)$

$$= (3,4) + (6,4)$$

$$= (9,8)$$

So $(\alpha+\beta).(\alpha_1,\alpha_2) \neq \alpha.(\alpha_1,\alpha_2) + \beta.(\alpha_1,\alpha_2)$

Thus $R^2(R)$ is not a vector space.

1.02 properties of vector space

Theorem 1 : Let V (F) be a vector space

then (i) c.0 = 0 Where c $\in$ F and 0 $\in$ V

(ii) 0. α = 0 where 0 $\in$ F and $\alpha \in$ V

(iii) (-c)α = - (cα) = c (-α) $\forall$ c $\in$ F and $\alpha \in$ V

(iv) c (α - β) = c α -c β $\forall$ c $\in$ F and α , $\beta \in$ V

Proof : (i) Consider c. α + c. 0 = c (α +0) =c. α = c. α + 0

$\therefore$ by left cancellation law, c.0 = 0

(ii) Consider c. α + 0. α = (c+0) . α = c. α +0. α

= c. α + 0 (from (i))

$\therefore$ c. α + 0. α = c. α + 0.

By left cancellation law , 0. α = 0

(iii) Consider c. α + (-c) α = [c+ (-c)] α

$= 0.\ \alpha = 0$ (from (ii))

and $(-c)\ \alpha + c\alpha = (-c + c)\ \alpha$

$= 0.\ \alpha = 0$ (from (ii))

$\therefore \quad c\alpha + (-c)\ \alpha = (-c)\ \alpha + c\alpha = 0$

$\therefore \quad (-c)\ \alpha$ is the inverse of c. α

$(-c)\ \alpha = -(c\alpha)$

$|||^{ly}$ $c(-\alpha) = -(c\alpha)$

$\therefore \quad (-c)\ \alpha = -(c\alpha) = c(-\alpha)$

(iv) consider $c(\alpha - \beta) = c[\alpha + (-\beta)]$

$= c\ \alpha + c(-\beta)$

$= c\ \alpha + (-c)\ \beta$

$= c\alpha - c\beta$

Theorem 2 : If V(F) is a vector space over a field F,

Then (i) $(-1)\alpha = -\alpha$

(ii) $\beta + (\alpha - \beta) = \alpha$

(iii) If a $\alpha = 0$ then either a =0 or $\alpha = 0$

Proof : (i) we have $(-c)\ \alpha = -(c\alpha)$

Take c = 1 $\therefore$ $(-1)\ \alpha = -(1.\ \alpha)$

ie,$(-1)\alpha = -\alpha$

(ii) $\beta + (\alpha - \beta) = \beta + [\alpha + (-\beta)]$

$= \beta + [-\beta + \alpha]$ $\because$ (V,+) is commutative

$= [\beta + (-\beta)] + \alpha$

$= 0 + \alpha = \alpha$

(iii) $a\alpha = 0$ (given)

Let $a \neq 0$. Then we shall show that $\alpha = 0$.

Since $a \in$ the field F and $a \neq 0$,

there exists $a^{-1} \in$ F such that $a^{-1}a = a.a^{-1} = 1$.

Now $\alpha = 1 .\ \alpha = (a^{-1}a)\ \alpha = a^{-1}(a\alpha) = a^{-1}(0) = 0$

Again if a $\alpha = 0$ and $\alpha \neq 0$ then we have a = 0

Otherwise, i.e, if $a \neq 0$. then as we have proved above,

$\alpha = 0$ which contradicts the assumption that $\alpha \neq 0$.

$\therefore \quad a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

Theorem 3: **If v(F) is a vector space then the cancellation laws hold .**

(i) $\mathbf{a\ \alpha = b\alpha \Rightarrow a=b\ ;\ \alpha \neq 0, a,b \in F}$

(ii) $\mathbf{a\alpha = a\beta \Rightarrow \alpha = \beta\ ;\ a \neq 0,\ \alpha,\ \beta \in v}$

***Proof* :** (i) $a\ \alpha = b\alpha \Rightarrow [a + (-b)]\ \alpha = \alpha + (-b)\ \alpha$

$\Rightarrow [a + (-b)]\ \alpha = [b + (-b)]\ \alpha$

$\Rightarrow [a + (-b)]\ \alpha = 0\alpha$

$\Rightarrow [a + (-b)] = 0$

$\Rightarrow a + (-b) = 0 \because \alpha \neq 0$ (given)

$\Rightarrow a = b.$

(iii) since $a \neq 0$,$\therefore$ a^{-1} exists in F such that

$a\ a^{-1} = a^{-1}a = 1.$

$\therefore\ a\alpha = a\beta \Rightarrow a^{-1}(a\ \alpha) = a^{-1}(a\beta)$

$\Rightarrow (a^{-1}a)\ \alpha = (a^{-1}a)\beta$

$\Rightarrow 1.\ \alpha = 1.\beta$

$\Rightarrow \alpha = \beta$

1. 03 subspaces

***Definition* : A non –empty subset w of avector v is said to be a subspace of v over a field F if W is a vector space over F w.r.t. the same operation as in V.**

***Example* :** The set V of all ordered triples (x_1, x_2, x_3) over the field of real numbers , is a vector space w.r.t addition and scalar multiplication . the set w of all ordered triplets of the form $(x_1, x_2, 0)$ isa subset of V and W is a subspace of V.

It is easy to verify that W satisfies all the vector space axioms . We shall see more example later.

***Note* :** (i) Every vector space has always two subspaces {0} and V. These are called trivial subspaces and other subspace is called a non-trivial subspace of V.

1.04 Criterion for a subset to be a subspace :

Theorem 1 : A non – empty subset W of a vector space V is a subspace of V if and only if

(i) $\alpha \in$ w , $\beta \in$ W $\Rightarrow \alpha + \beta \in$ W and

(ii) c$\in$ F, $\alpha \in$ W $\Rightarrow$ c $\alpha \in$ W

***Proof* :** (a) Let W be a vector space over a field F

$\therefore$ W satisfies all the vector space axioms .

$\therefore$ W is closed w.r.t. the addition and scalar multiplication .

$\therefore \forall\ \alpha , \beta \in$ W, $\alpha + \beta \in$ W.

and $\forall\ \alpha \in$ W and c$\in$ F, c.$\alpha \in$ W

(b) Conversely, let W be a non-empty subset of V such that the condition (i) and (ii) are satisfied. We have to prove that w is a subspace of V, thus, we have to prove that W satisfies all the vector space axioms.

(V_1) (a) $\alpha + \beta \in$ W, $\forall\ \alpha , \beta \in$ W.

(b) Since $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ is satisfied in V, it is satisfied in the subset W also .

(c) $\forall\ \alpha \in$ W, c. $\alpha \in$ W. take c = -1

$\therefore$ $\forall\ \alpha \in$ W, (-1) α = -$\alpha \in$ W.

$\therefore$ from condition (i) , α + (-α)$\in$ W i.e, 0 $\in$ W.

(d) $\forall\ \alpha \in$ W, where exists c = -1 $\in$ F such that

cα = (-1) α = - $\alpha \in$ W

(e) $\alpha + \beta = \beta + \alpha$ is satisfied in V, hence it is satisfied in the subset W also .

(V_2) c ($\alpha + \beta$) = cα +cβ

and (c_1+c_2) α = $c_1\alpha + c_2\alpha$ are satisfied in V.

Hence they are satisfied in the subset W also.

(V_3) (c_1c_2) α = $c_1(c_2\alpha)$ is satisfied in V and hence it is satisfied in the subset W also.

(V_4) 1. α = α .$\forall\ \alpha \in$ W and 1$\in$ F .

Theorem 2 : A non-empty subset W of a vector space V is a subspace of v if and only if

(i) $\alpha, \beta \in W, c_1, c_2 \in F \Rightarrow c_1\alpha + c_2\beta \in W$

***Proof* :** Since W is closed w.r.t scalar multiplication , $c_1\alpha$ and $c_2\beta \in$ W. and since W is closed w.r.t addition , $c_1\alpha + c_2\beta \in$ W.

$\therefore$ If W is a subspace of V. then $c_1\alpha + c_2\beta \in$ W.

and as $c_1\alpha + c_2\beta \in$ W.

choose c_1= 1, c_2= 1 $\therefore$ 1α +1$\beta \in$ W.

$\Rightarrow$ $\alpha + \beta \in$ W

and $\forall c \in F, C\alpha \in W$

$\therefore$ From the previous theorem (the necessary and sufficient condition), it follows that W is a subspace of V.

$\therefore$ W is a subspace of V iff $c_1\alpha + c_2\beta \in$ W.

Note : whenever we have to prove that W is a subspace of V, it is enough to verify that W is a non-empty subset of V and

$\forall\ \alpha, \beta \in W, c_1, c_2 \in F \Rightarrow c_1\alpha + c_2\beta \in W$

Theorem 3 : The intersection of two subspaces of a vector Space V over a field F is a subspace of V.

***Proof* :** Let Sand T be two subspaces of V

S $\cap$ T = { α | $\alpha \in$ S and α $\in$ T}.

We have to prove that $\forall\ \alpha , \beta \in$ S $\cap$ T,

$c_1\alpha + c_2\beta \in$ S $\cap$ T where $C_1, C_2 \in$ F .

$\alpha + \beta \in$ S $\cap$ T $\Rightarrow \alpha, \beta \in$ S and $\alpha, \beta \in$ T

$\Rightarrow$ $c_1\alpha + c_2\beta \in$ S and $c_1\alpha + c_2\beta \in$ T

since S and T are subspaces.

$\Rightarrow$ $c_1\alpha + c_2\beta \in$ S $\cap$ T .

$\therefore$ S $\cap$ T is a subspaces of V

***Note* :** (i) This result can be extended to any finite number subspaces.

i.e $S_1 \cap S_2 \cap \ldots\ldots\ldots\ldots \cap S_n$ is a subspaces of V.
Whenever $S_1, S_2 \ldots\ldots\ldots S_n$ are subspaces of V.

(ii) The union of any two subspaces of V need not be a subspace of V.
For eg . In V_2 (R) let $S = \{ (X, 0) \mid X \in R\}$ and $T = \{ (0,Y) \mid Y \in R\}$ be two subspaces.

Then $S \cup T = \{ \alpha \mid \alpha \in S \text{ or } \alpha \in T\}$

Let $(x, 0), (0,y) \in S \cup T$

Then $(x,0) + (0,y) = (x,y) \notin S \cup T$. $\forall x,y \in R$.

For eg . $(1,0) + (0,1) = (1,1) \notin S \cup T$.

$\therefore$ Closure axiom is not satisfied w.r.t .+

$\therefore$ W is not a subspace of V_2 (R)

Theorem 4 : The union of two subspaces of a vector space V over a field F is a subspace if and only if one is contained in the other.

Proof : Let S and T be two subspaces of V(F)
Let $S \cup T$ be a subspace of V(F)
T.P.T : $S \subseteq T$ or $T \subseteq S$

If $S \not\subset T \Rightarrow \exists \alpha S ; \alpha \notin T$

and $T \not\subset S \Rightarrow \exists \beta T : \beta \notin S$

$\Rightarrow \alpha \in S \Rightarrow \alpha \in S \cup T$

$\beta \in T \Rightarrow \beta \in S \cup T$

Since $S \cup T$ is a subspace of V(F)

$\alpha, \beta \in S \cup T \Rightarrow C_1 \alpha + C_2 \beta \in S \cup T \quad \forall C_1, C_2 \in F$

$\Rightarrow \beta \in S$ and $\alpha \in T$

which is a contradiction

$\therefore$ our assumption is wrong

$\Rightarrow S \subseteq T$ or $T \subseteq S$

Conversely. Let S and T be two subspaces of V such that $S \subseteq T$ or $T \subseteq S$

$\Rightarrow S \cup T = S$ or $S \cup T = T$

$\therefore$ $S \cup T$ is also a subspace of V(F)

Worked examples :

(1) Prove that the subset $W=\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ of the vector space V_3(R) is a subspace of V_3(R).

Solution : W is a non – empty subset of V_3(R)

Let $\alpha, \beta \in W$ and $c_1, c_2 \in R$.

$\therefore$ $\alpha = (x_1, x_2, x_3)$ such that $x_1+x_2+x_3 = 0$

and $\beta = (y_1, y_2, y_3)$ such that $y_1+y_2+y_3 = 0$

$\therefore$ $c_1\alpha + c_2\beta = c_1(x_1,x_2,x_3) + c_2(y_1,y_2,y_3)$
$= (c_1x_1+c_2y_1, c_1x_2+c_2y_2, c_1x_3+c_2y_3)$

and $c_1x_1+c_2y_1+c_1x_2+c_2y_2+c_1x_3+c_2y_3$
$= c_1(x_1,x_2,x_3) + c_2(y_1,y_2,y_3) = c_1(0) + c_2(0) = 0.$

$\therefore$ $c_1\alpha + c_2\beta \in W$

$\therefore$ W is a subspace of V_3(R).

(2) Prove that the subset $W = \{ (x,y,z,) \mid x\text{-}3y + 4z = 0\}$ of the vector space R^3 is a subspace of R^3.

Solution **:** W is a non empty subset of R^3 since at least one element $(0,0,0) \in W$. Such that $0\text{-}3.0+ 4.0 = 0$.

Let $\alpha, \beta \in W$ and $c_1, c_2 \in R$.

$\therefore$ $\alpha = (x_1, y_1, z_1)$ such that $x_1 - 3y_1 + 4z_1 = 0$

$\beta = (x_2, y_2, z_2)$ such that $x_2 - 3y_2 + 4z_2 = 0$

$\therefore$ $c_1\alpha + c_2\beta = c_1(x_1,y_1,z_1) + (x_2,y_2,z_2)$
$= (c_1x_1+c_2x_2, c_1y_1+c_2y_2, c_1z_1+c_2z_2)$

and $c_1x_1+c_2x_2-3(c_1y_1+c2y2)+4c1z1+c_2z_2)$
$= c_1(x_1-3y_1+4z_1) + c_2(x_2-3y_2+4z_2)$
$= c_1(0) + c_2(0) = 0$

$\therefore\ c_1\alpha + c_2\beta \in W$.

(3) Prove that the subset W = { (x, y, z)| x = y = z } is a subspace of V^3(R)

***Solution* :** W is a non-empty subset of V^3(R)

Let $\alpha, \beta \in W$

$\therefore\quad \alpha = (x_1, y_1, z_1)$ such that $x_1 = y_1 = z_1$

$\beta = (x_2, y_2, z_2)$ such that $x_2 = y_2 = z_2$

$c_1\alpha + c_2\beta = c_1(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (c_1x_1+c_2x_2, c_1y_1+c_2y_2, c_1z_1+c_2z_2)$

$x_1 = y_1 = z_1$

and $x_2 = y_2 = z_2$ } $\Rightarrow$ c_1x_1+c2x2=c1y1+c_2y_2=$c_1z_1+c_2z_2$

$\therefore\quad c_1\alpha + c_2\beta \in W$

$\therefore$ W is a subspace of V_3(R).

(4) If a vector space is the set of real valued continous fuctions over the field of real numbers, then prove that the set w of solutions of the differential equation $2\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 2y = 0$ is a subspace of V.

***Solution* :**

$$W = \left\{ 2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0 \right\}$$

Clearly y = 0 satisfies the given differential equation .

$\therefore$ 0 $\in$ W and hence W is non-empty.

Let y_1 and $y_2 \in W$ and $c_1, c_2 \in R$ then we have to show that $c_1y_1+c_2y_2$ satifies the differential equation.

Consider $2\dfrac{d^2}{dx^2}(c_1y_1+c_2y_2) - 9\dfrac{d}{dx}(c_1y_1+c_2y_2) + 2(c_1y_1+c_2y_2)$

$$= c_1\frac{d^2y_1}{dx^2} + 2c_2\frac{d^2y_2}{dx^2} - 9\frac{dy_1}{dx} - 9c_2\frac{dy_2}{dx} + 2c_1y_1 + 2c_2y_2$$

$$= c_1\left(2\frac{d^2y_1}{dx^2} - 9\frac{dy_1}{dx} + 2y_1\right) + c_2\left(2\frac{d^2y_2}{dx^2} - 9\frac{dy_2}{dx} + 2y_2\right)$$

$= c_1(0)+c_2(0) \because y_1, y_2$ satisfy the given differential equation.

$= 0$.

$\therefore\quad c_1y_1+c_2y_2 \in W$

$\therefore$ W is a subspace of V.

(5) Verify whether W = $\left\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \le 1\right\}$ of the V_3(R) is a subspace of V_3(R).

***Solution* :**

Consider α = (1, 0,0) , β = (0.0,1)

Clearly $\alpha, \beta \in W$ since $1^2 + 0^2 + 0^2 \le 1$and

$0^2 + 0^2 + 1^2 \le 1$

Now $\alpha + \beta$ = (1,0,0) + (0,0,1) = (1,0,1)

and $1^2 + 0^2 + 1^2 = 2$ which is not less than or equal to 1.

$\therefore$ (1,0,1) $\in$ W

ie, $\alpha + \beta \notin W$ $\therefore$ W is not a subspace of V.

(6) Examine the subset V = {(a + 2b, a, 2a – b, b | a, b $\in$ R)} for a subspace of R^4

***Solution* :** Let $\alpha = (a_1 + 2b_1, a_1, 2a_1 - b_1, b_1)$

$\beta = (a_2 + 2b_2, a_2, 2a_2 - b_2, b_2)$ are in V

Consider $c_1\alpha + c_2\beta = c_1(a_1 + 2b_1, a_1, 2a_1 - b_1, b_1)$

$+ c_2(a_2 + 2b_2, a_2, 2a_2 - b_2, b_2)$

From the addition of order pair in R^4,

$= (c_1(a_1 + 2b_1, c_1a_1, c_1(2a_1 - b_1), c_1b_1)$

$+ (c_2(a_2 + 2b_2), c_2a_2, c_2(2a_2 - b_2), c_2b_2)$

$= (c_1(a_1 + 2b_1 + c_2(a_2 + 2b_2), c_1a_1 + c_2a_2, c_1(2a_1 - b_1) + c_2(2a_2 - b_2),$

$c_1 b_1 + c_2 b_2) \in V$

$\therefore$ V is a sub space of R^4

EXERCISE

(1) Show that the set V of all ordered pairs of integers does not form a vector space over the field R of reals.

(2) Show that the set of all pairs of real numbers over the field of reals define as $(x_1 y_1)+(x_2 y_2)=(3y_1+3y_2,-x_1-x_2)$ and $c(x_1 y_1)=(3cy_1-cx_1)$ does not from a vector space.

(3) Let $V=\{(x,y) \mid x,y\in R\}$ and field is the set of reals show that V is not a vector space under + and scalar multiplication defined as in each of the following cases

(i) $(x,y)+(s,t)=(0,y+t), k(x,y)=(kx,ky)$

(ii) $(x,y)+(s,t)=(x+s,\ y+t), K(x,y)=(0,ky)$

(iii) $(x,y)+(s,t)=(x+s,y+t), k(x,y)=(kx,y)$

(4) Verify whether the following sets from vector spaces w.r.t the given operation and the given field.

(i) the field of complex numbers over the field of complex numbers

(ii) the field of complex numbers over field of real numbers .

(iii) the field of real numbers over the field of complex numbers w.r.t the usual addition and multiplication.

(5) Examine whether the set V of all orderd pair of integers from a vector space over the field R of real numbers , w.r.t . addition of ordered pairs and scalar multiplication of an ordered pair.

(6) Verify the following for a vector space : The set of all polynomials with real co efficient over the field of real numbers w.r.t . addition of polynomials and scalar multiplication of polynomials.

(7) Prove that V $=\left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a,b\in R\right\}$ is a vector space over the field of real numbers w.r.t addition of matrices and scalar multiplication of a matrix.

(8) Prove that every field F can be considered as a vector space over F w.r.t the operation in F.

(9) Prove that the set of all polynomials over the field of real numbers is a vector space w.r.t. the addition of polynomials and scalar multiplication of polynomial

(10) Prove that the set V = $\{x+y\sqrt{2}x, y\in Q\}$ where Q is the field or rationales ,w.r.t . addition and multiplication of real numbers, is a vector space.

(11) a) Prove that the set of all m x n matrices with real elements is a vector space over the field of real numbers w.r.t addition and scalar multiplication of matrices.

b) Show that the set of all matrices of the order $n \times n$ with their elements as real numbers is a vector space over the $\langle R+\ \cdot\rangle$ with the usual operations of matrices.

c) Show that the set of polynomials $\{0,1,2,\ x+1,\ x+2,\ 2x+1,\ 2x+2,\ x,\ 2x\}$ forms a vector space over the field $(I_3,\ +_3 \times_3)$ assuming the usual operations for polynomials.

(12) Verify which of the following are subspaces:

(i) S = $\{ (a_1,0,a_2) \mid a_1,a_2\in R\}$ of $V_3(R)$

(ii) W = { (x,2y,3z) : x,y,z $\in$ R} of V_3(R)
(iii) S= { (x,x,x) : x $\in$ R} of V_3(R)
(iv) W = { (x,y,z) : x,y,z$\in$ Q }of V_3
(v) S = { (x,y,z) : 2x+3y+ z = 0 of V_3(R)
(vi) W= { (x,y,z) : x =y } of V_3(R)
(vii) S = { (x,y,z) : xy = 0 } of V_3(R)
(viii) S = { (x,y,z) | $\sqrt{2}$ X = $\sqrt{5}$ y} of V_3(R)

(13) Which of the following sets are subspaces of the vector space V of all polynomials over the field of reals.
(i) The set of all polynomials of degree 4
(ii) The set of all polynomials of degree ≤ 4
(iii) The set of all polynomials of degree ≤ 5
(iv) The set of all polynomials of degree 5.

(14) Which of the following are subspaces of the vector spaces of all real valued continuous factions defined on [0,1], over R
(i) all function f for which 2f (0) = f(1)
(ii) all function f for which f (x) = 0, $\forall x \in [0,1]$
(iii) all function f for which f (x) = 1.

(15) Show that $w = \{(x,0,0) | x \in R\}$ is a subspace of R^3 over the field Reals R.

(16) Show that any plane passing through the origin is a subspace if V_3(R).

(17) Prove that w= {(x, y, 0) | x $\in$ R} is a subspace of R^3 over the field of Reals R.

(18) Determine whether or not the following subjects of R^4 are Sub space.

(i) $A = \{(a,b,c,d) | a+b = c+d\}$

(ii) $B = \{(a,b,c,d) \ | a+c = b+d\}$

(iii) $C = \{(a,b,c,d) \ | ab = cd\}$

(iv) $D = \{(a,b,c,d) \ | a^2 + b^2 = 0\}$

(v) $E = \{(2a+b, 2a-b, 0, c) | \ a,b,c \in R\}$

(19) Show that the following sub-sets are sub-spaces in $v_3(R)$

(i) $A = \{(0,b,c) | 0,b,c \in R\}$

(ii) $B = \{(a,b,c) | a-3b+4c = 0 \quad \forall a,b,c \in R\}$

(iii) $C = \{(x,y,z) | (x+2y), y, -x+3y), \quad \forall \ x,y,z \in R\}$

(iv) $D = \{(x,y,z) | x+y+2z = 0, \quad \forall \ x,y,z \in R\}$

(20) Prove that the set of all solutions (x_1, y_1, z_1) of the equation $x+3y+2z = 0$ is a subspace of the vector space $v_3(R)$

(21) Let $v = s$ and W be the set of all ordered triplets (x,y,z) such that $x-3y+4z = 0$. Prove that W is a subspace of R^3.

(22) If a vector space is the set of real valued continuous functions over the field of real numbers, then prove that the set W of solutions of the differential equation $y'' - 4y' + 3y = 0$ is a subspace of V.

Answers

(4) (i) yes, (ii) yes, (iii) no (5) no (6) yes

(12) (i) yes, (ii) yes, (iii)yes (iv) no (v) yes (vi) yes (vii)no(viii)yes

(13) (i) no, (ii) yes, (iii) yes, (iv) no.

(14) (i) yes,(ii) yes,(iii) yes

(18) (i), (ii), (v) yes

1.05 Linear span of a set

Definition : Let V be a vector space over the field F. and $\alpha_1, \alpha_2, \ldots\ldots \alpha_m$ be any m vectors of V. Any vector of the form $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_m\alpha_m$ where $c_1, c_2, \ldots\ldots\ldots c_m \in F$ is called a *linear combination* of the vectors $\alpha_1, \alpha_2 \ldots\ldots\ldots \alpha_m$.

Definition : Let V be a vector space over the field F. and S be any non-empty subset of V. Then the linear span of S is the set of all linear combination of any finite number of elements of S and is denoted by L(S)

$\therefore L(S) = \{ C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots\ldots + C_m\alpha_m : \alpha_i \in S \text{ and } c_i \in F\},$
$i = 1, 2, \ldots..m.$

Theorem 1 : Let $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_m$ be m vectors of a vectors space V over the field F. Then the set of all linear combinations of $\alpha_1, \alpha_2, \ldots\ldots \alpha_m$ is a subspace of V and it is the smallest subspace containing all the given vectors.

Proof **:** Let S be the set of linear combinations of $\alpha_1, \alpha_2, \ldots\ldots \alpha_m$
ie.,S $=\{\alpha \,|\, \alpha = C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots + C_m\alpha_m ; C1 \in F\}, i = 1,2,\ldots.m$
Now S is non-empty $\because$ every α_1 can be written as 1. α_1 and hence $\alpha_1 \in$ S
Let $\alpha, \beta \in S$

$\therefore \quad \alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots\ldots + c_m\alpha_m$
$\beta = \quad d_1\alpha_1 + d_2\alpha_2 + \ldots.d_m\alpha_m$ where $c_i d_i \in F$.
$\therefore \alpha + \beta = (c_1+d_1)\,\alpha_1 + (c_2+d_2)\alpha_2 + \ldots\ldots + (c_m+d_m)\,\alpha_m$
= a linear combination of $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_m$
$\therefore \alpha + \beta \in S$

$c\alpha = (cc_1)\,\alpha_1 + (cc_2)\alpha_2 + \ldots\ldots\ldots\ldots + (cc_m)\alpha_m$
= a linear combination of α_i.

$\therefore$ $c\,\alpha \in$ S.

$\therefore$ S is closed w.r.t. scalar multiplication.

$\therefore$ S is a subspace of V.

Now ,we shall that S is the smallest subspace containing $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_m$

Let W be any other subspace of V such that $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_m \in W$

We shall show that $S \subset W$

Let $\alpha = C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots\ldots\ldots + C_m\alpha_m \in S$

$\therefore \quad \alpha \in W$

$\therefore \quad S \subset W$

ie ,S is the smallest subspace containing $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_m$.

Note :- (1) The subspace of all linear combination of the set of given vector space is called the subspace generated by these vectors or spanned by these vectors .
(2) The subspace spanned by any nonzero vector α of a vector space V, consists of all scalar multiples of α. Geometrically ,it represents a line through the origin and α.
(3) The subspace spanned by any two non zero vectors α and β, which are not multiples of each other represents a plane passing through the origin, α and β.

Worked Examples

(1) Express the vector (2,-1,-8) as a linear combination of the vectors (1,2,1), (1,1,-1), (4,5,-2).

Solution **:** (2,-1,-8) = a (1,2,1) + b (1,1,-1) +c (4,5 , -2)
= (a,2a,a) + (b,b-b) + (4c,5c, -2c)
= (a+b+4c ,2a + b +5c ,a- b-2c)

$\therefore$ a + b + 4c = 2 --------(1)
2a + b +5c = -1 --------(2)

a - b –2c = -8 --------(3)

Solving these simultaneous equations for a,b, c we get

(1) – (2) $\Rightarrow$ -a – c =3 or a +c = - 3

(2) + (3) $\Rightarrow$ 3a + 3c or a + c = - 3.

Now giving some value for c, say , c = 1, we get a = -4.
Substituting these values of a and c in (1) weget

-4 + b + 4 = 2 $\therefore$ b = 2

$\therefore$ (2, -1,-8) = - 4 (1, 2, 1) + 2 (1, 1, -1) + 1 (4,5,-2)

***Note* :** The linear combination is not unique since by choosing c = 0, we get a = -3 and b = 5.

(2) Prove that (3,-7,6) is in the span of the vectors (1, -3,2),(2,4,1),(1,1,1).

***Solution* :** To prove that (3, -7,6) is in the span of the vectors (1, -3,2),(2,4,1),(1,1,1), we have to express (3, -7,6) as a linear combination of these.

(3, -7,6) = a (1, -3,2) +b(2,4,1) + c (1,1,1)

= (a + 2b + c , -3a +4b + c , 2a +b + c)

$\therefore$ a + 2b + c = 3 ------------(1)

-3a +4b + c = -7 -----------(2)

2a +b + c = 6 ----------(3)

(1) – (2) $\Rightarrow$ 4a –2b = 10 $\Rightarrow$ 2a-b = 5……..(4)

(2) – (3) $\Rightarrow$ -5a +3b = - 13 ……(5)

Multiple (4) by 3 $\therefore$ 6a –3b = 15 ………(6)

(3) + (6) $\Rightarrow$ a = 2

$\therefore$ (4) $\Rightarrow$ 4 - b = 5 $\Rightarrow$ b = -1

(1) $\Rightarrow$ -2 +c = 3 $\Rightarrow$ c = 3.

$\therefore$ (3,-7 ,6) = 2 (1,-3,2) –1 (2,4,1) + 3 (1,1,1)

(3) Which of the vectors (1,-3,5) and $\left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right)$ is in the span of (1,2,1), (1,1-1) and (4,5,-2).

***Solution* :** (1,-3,5) = a (1,2,1) + b (1,1,-1) + c (4,5,-2)

= (a +b+4c+,2a + b +5c,a-b-2c)

$\therefore$ a +b + 4c = 1 . . (1)

2a + b + 5c = - 3 . . . (2)

a –b – 2c = 5 . . (3)

(1) –(2) $\Rightarrow$ -a – c = 4 $\Rightarrow$ a + c = - 4 . . (4)

(2) + (3) $\Rightarrow$ 3a +3c = 2 . . . (5)

$\Rightarrow a + c = \frac{2}{3}$ . . . (6)

The equation are inconsistent since (4) contradicts (6)

$\therefore$ There do not exit scalars a,b, c such that (1,-3,5) is expressed as a linear combination of (1,2,1) , (1,1,-1),(4,5,-2).

$\therefore$ (1, -3,5) does not lie in the span of the given vectors.

Consider $\left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right)$ = a (1,2,1) + b (1,1,-1) + c (4,5,-2)

= (a +b+4c+,2a + b +5c,a-b-2c)

$\therefore$ a +b + 4c = $-\frac{1}{3}$ $\Rightarrow$ 3a + 3b + 12c = - 1 ………..(1)

2a + b + 5c = $-\frac{1}{3}$ $\Rightarrow$ 6a + 3b + 15 c = - 1 ………(2)

a –b – 2c = $\frac{1}{3}$ $\Rightarrow$ 3a – 3b – 6c = 1 ………..(3)

(1) - (2) $\Rightarrow$ - 3a – 3c = 0 $\Rightarrow$ a +c = 0.

(2) + (3) $\Rightarrow$ 9a +9c = 0 $\Rightarrow$ a + c = 0.

Choose c = 1 , $\therefore$ a = - 1

Substituting these in (1) we get

-3 + 3b + 12 = -1

3b = -10

$\therefore$ b = $-\frac{10}{3}$

$\therefore \left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right) = -1\,(1,2,1) - \frac{10}{3}(1,1,-1) + (4,5,-2)$

$\therefore \left(\frac{-1}{3},\frac{-1}{3},\frac{1}{3}\right)$ lies in the span of the vectors (1,2,1), (1,1,-1) ,(4,5,-2).

(4) Verify whether $\begin{bmatrix}3 & -1\\ 1 & -2\end{bmatrix}$ is in linear span of

$$\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix},\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix},\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

Solution :

$$\begin{pmatrix}3 & -1\\ 1 & -2\end{pmatrix} = a\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}+b\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}+c\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}a+b+c & a+b-c\\ 0-b+c & -a+0+0\end{bmatrix}$$

$$= \begin{bmatrix}a+b+0 & a+b-c\\ -b & -a\end{bmatrix}$$

$\therefore$ a + b +c = 3 ,a + b – c = - 1
- b = 1, - a = - 2

$\therefore$ b = -1 , a = 2, $\therefore$ c = 2

Hence $\begin{bmatrix}3 & -1\\ 1 & -2\end{bmatrix} = 2\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}-1\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}+2\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$

$\therefore \begin{bmatrix}3 & -1\\ 1 & -2\end{bmatrix}$ lies in the span of the given vectors.

(5) Prove that the xy – plane is spanned by the vectors (1,2,0) and (0,1,0) in R^3

***Solution* :** xy – plane is { (a,b,0) |a,b ∈ R}
(a,b,0) = k_1(1,2,0) + k_2 (0,1,0)

= (k_1,2k_1,+ k_2,0)
$\therefore$ k_1 = a, 2k_1 + k_2= b.
$\Rightarrow$ 2a + k_2 = b
$\Rightarrow$ k_2 = b – 2a.

$\therefore$ (a,b,0) = a (1,2,0) + (b-2a) (0,1,0)
Hence (a,b,0) is expressed as linear combination of (1,2,0) and (0,1,0).
$\therefore$ xy- plane is spanned by (1,2,0) and (0,1,0).

(6) Find the subspace spanned by the vectors (3,0,0) and (0,0,-5) of the vector space V_3(R)

***Solution* :** Any vector subspace S is of the form
a(3,0,0) + b (0,0,-5) where a,b ∈ R
= (3a,0,0) + (0,0,-5b)
= (3a, 0,-5b)
$\therefore$ S = { (3a, 0,-5b) : a,b ∈ R }.

(7) For what value of k, the vector (1, k, 5) is a linear combination of vectors (1, -3, 2) and (2, -1, 1)

***Solution* :** Let (1, k, 5) = C_1(1, -3, 2) + C_2 (2, -1, 1)
= (C_1 + 2C_2, -3C_1,-C_2, 2C_1 + C_2)

Equating the respective components

$C_1 + 2C_2 = 1$ . . . (1)

$-3C_1 - C_2 = k$ . . . (2)

$2C_1 + C_2 = 5$ . . . (3)

From (1) - 2(3) $\Rightarrow$ $C_1 + 2C_2 = 1$

$4C_1 + 2C_2 = 10$

$-3C_1 = -9$ $\Rightarrow$ $\therefore c_1 = 3$

Substituting C_1 = 3 in (1) we get C_2 = -1
If system of equations are consistent then

$-3(3) - (-1) = k$

$k = -8$

(8) Write the vector $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$**, in vector space 2 x 2 matrices as a linear combination of** $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

***Solution* :** Let $A = C_1 B + C_2 C + C_3 D$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = C_1 \; C_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_1 + C_2 + C_3 & C_1 + C_2 - C_3 \\ -C_2 & -C_1 \end{bmatrix}$$

Equating the respective elements

$1 = -C_2, -2 = -C_1, \quad 3 = C_1 + C_2 + C_3$

$\therefore C_2 = -1, \quad C_1 = 2 \quad C_3 = 2$

$\therefore$ Given vector A is a linear combination with B, C, D for the above constants. i.e.,

$A = 2 - C + 2D.$

(9) Show that $3x^2 + x + 5$ polynomial is the linear span of the set S= {x^3, $x^2 + 2x$, $x^2 + 2$, $1 - x$}

Let $3x^2 + x + 5 = c_1(x^3) + c_2(x^2 + 2x) + c_3(x^2 + 2) + c_4(1-x)$

$= c_1(x^3) + x^2(c_2 + c_3) + x(2c_2 - c_4)$

$+ 2c_3(1-x) + c_4$

Equating the respective degree terms,

$c_1 = 0, \; c_2 + c_3 = 3, \; 2c_2 - c_4 = 1, \quad 2c_3 + c_4 = 5$

Solving : $c_2 = 3, \quad c_3 = 0, \quad c_4 = 5$

$\therefore 3x^2 + x + 5 = 3(x^2 + 2x) + 5(1 - x) \in L(S)$

(10) Let $\alpha = (1, 2, 1)$ $\beta = (3, 1, 5)$ and $\gamma = (-1, 3, -3)$ of $V_3(R)$. Show that {α, β} {α, β, γ} are the same subspaces of V_3®

***Solution*:** Let $T = \{\alpha, \beta\}$ $\quad S = \{\alpha, \beta, \gamma\}$

Since $T \subset S$, we have $L(T) \subset L(S)$

Let $\delta \subset L(S) \quad \Rightarrow \quad C_1 \alpha + C_2 \beta + C_3 \gamma = \delta \quad \ldots (1)$

$\gamma \subset T(T) \quad \Rightarrow \quad a_1 \alpha + b_2 \beta = \gamma$

$a_1(1, 2, 1) + b_1(3, 1, 5) = (-1, 3, -3)$

$\Rightarrow a_1 + 3b_1 = -1$

$2a_1 + b_1 = 3$

$a_1 + 5b_1 = -3$

Solving the above, $a_1 = 2, \quad b_1 = -1$

Thus $2\alpha - \beta = \gamma$

(1) $\Rightarrow \quad C_1 \alpha + C_2 \beta + C_3(2\alpha - \beta) = \gamma$

$(C_1 + 2C_3)\alpha + (C_2 - C_3)\beta = \gamma$

$\therefore \delta$ is a Linear combination of the set S

$\therefore L(S) \subset L(T)$

$\therefore L(S) = L(T)$

Exercises

1. Express the vector (1, -2, 5) as a linear combination of the vectors (1, 1, 1) (1, 2, 3), (2, -1, 1).
2. Prove that (2, -5, 4) can not be expressed as a linear combination of (1, -3, 2) and (2, -1, 1)
3. Write the vector (1, 7, -4) as a linear combination of vectors $\alpha_1(1, -3, 2)$ and $\alpha_2(2, -1, 1)$ vector space $V_3(R)$.

4. Is the vector $\alpha = (2, -5, 3)$ in $V_3(R)$ a linear combination of vectors $\alpha_1 = (1, -3, 2)$ $\alpha_2 = (2, -4, -1)$ $\alpha_3 = (1, -5, 7)$
5. Show that the vector $\alpha = (2, 2, 3)$ is in the span of the vectors $\alpha_1 = (2, 1, 4)$ $\alpha_2 = (1, -1, 3)$ and $\alpha_3 = (3, 2, 5)$
6. a) Find K so that (1, K, 5) is a linear combination of (1, -3, 2) and (2, -1, 1)
   b) For what value of K will the vector (1, -2, K) be a linear combination of the vectors (3, 0, -2) & (2, -1, -5)
7. Prove that the xz-plane may be spanned by the vectors (3, 0, 1) and (-3, 0, 2)
8. In R^3 show that the plane X = 0 may be spanned by the vectors (0, 2, 2) and (0, 4, 1).
9. Express (3,-7,6) as a linear combination of the vectors (1,-3,2), (2,4,1), (1,1,1) in $V_3(R)$.
10. Express (-1 ,4,-4) as a linear combination of the vectors (3,0,4) and (-2,2,-4) in R^3
11. Can the vectors (3,1,4) be expressed as a linear combination of (2,3,1) and (1,2,3)?
12. Examine whether the vectors (i) (3,3,3,), (ii) (4,2,6) (iii) (1,5,6), (iv) (0,0,0) can be expressed as a linear combination of the vectors (1,-1,3) and (2,4,0).
13. Which of the following are linear combination of

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, B\begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix}, C\begin{pmatrix} 4 & -2 \\ 0 & -2 \end{pmatrix}$$

$$\text{(a)} \begin{pmatrix} 6 & 3 \\ 0 & 8 \end{pmatrix}, \text{(b)}\begin{pmatrix} -1 & 7 \\ 5 & 1 \end{pmatrix}, \text{(c)}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{(d)} \begin{pmatrix} 6 & -1 \\ -8 & -8 \end{pmatrix}$$

14. Which of the following sets span $V_3(R)$.
    a) { (1,1,1),(2,2,0),(3,0,0)
    b){ (2,-1,3),(4,1,2),(8,-1,8)}
    b) { (1,2,1),(2,1,0),(1,-1,2)}
    c) { (1,0,0),(1,1,0), (1,1,1),(0,1,0)}

15. Express the following as linear combination of
    $P_1 = 4x^2 + x + 2$,
    $P_2 = 3x^2 - x + 1$, P_3 $5x^2 + 2x + 3$

(a) $5x^2 + 9x + 5$, (b) $6x^2 + 2$,
(c) $3x^2 + 2x + 2$, (d) 0.

Answers

1. $(1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) - (2, -1, 1)$
3. $(1, 7 -4) = -3\alpha_1 + 2\alpha_2$
4. No. 6. a) K = -8 b) K = -8

9. 2 (1,-3,2) + (-1) (2,4,1)+ 3(1,1,1)

10. 1(3,0,4) + 2(-2,2,-4)

11. N0

12. (i), (ii), (iv) are expressible

13. (a) , (c), (d) are linear combinations

14 (a) yes, (b) no, (c) yes, (d) yes

15 (a) $3p_1 - 4p_2 + p_3$ (b) $4p_1 - 2p_3$
(c) $\frac{1}{2}(p_1 - p_2 + p_3)$ (d) $O(p_1 + p_2 + p_3)$

1.06 Linear Independence and Dependence:

Definition **:** A set $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ of vectors of a space V over a field F said to be linearly independent if there exist scalars c_1, c_2 , c_n such that $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n = 0$ then $c_1 = 0$, $c_2 = 0 \ldots\ldots.. c_n = 0$

Definition **:** A set $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ of vectors of a vector space V over a field F is said to be *linearly dependent* if it is not linearly independent. i.e., the set $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ of a vector space over a field F is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots\ldots\ldots, c_n$ not all zero such that $c_1\alpha_1 + c_2\alpha_2 \ldots\ldots + c_n\alpha_n = 0$

***Note* :** The null set ϕ is always taken as linearly independent set.

Worked Examples:

(1) Show that the set S = {(1,0,0)}, (0,1,0), (0,0,1)} is linearly independent in $V_3(R)$.

***Solution* :** Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Consider $c_1e_1+c_2e_2+c_3e_3 = 0$

$\Rightarrow c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$

$\Rightarrow (c_1,0,0)+(0,c_2,0)+c_3(0,0,c_2)=(0,0,0)$

$\Rightarrow (c_1,c_2,c_3)=(0,0,0)$

$\Rightarrow c_1=0,c_2=0,c_3=0$

$\therefore S =\{e_1,e_2, e_3\}$ is linearly independent.

(2) Show that the set S {(1,1,1),((2,2,0), (3,0,0)} is linearly independent.

***Solution* :** Let $\alpha = (1,1,1)$, $\beta = (2,2,0)$, $\gamma = (3,0,0)$

$\therefore c_1\alpha+c_2\beta+c_3\gamma=0$

$\Rightarrow c_1(1,1,1)+c_2(2,2,0)+c_3(3,0,0)=(0,0,0)$

$\Rightarrow (c_1+2c_2+3c_3,c_1+2c_2, c_1)=(0,0,0)$

$\Rightarrow c_1+2c_2+3c_3=0,c_1+2c_2=0,c_1=0$

$c_1=0,c_1+2c_2=0 \Rightarrow 2c_2=0,$

$c_1=0,c_2=0,c_1+2c_2+3c_3=0 \Rightarrow c_3=0$

$\therefore c_1=0,c_2=0,c_3=0$

$\therefore S=\{\alpha, \beta, \gamma\}$ is linearly independent

(3) Prove that the set S = {(1,3,2), (1,-7,-8), (2,1,-1)} is linearly dependent.

***Solution* :** Let $\alpha = (1,3,2)$, $\beta = (1,7-8)$, $\gamma = (2,1-1)$

$\therefore c_1\alpha+c_2\beta+c_3\gamma = 0$

$\Rightarrow c_1(1,3,2)+c_2(1,-7,-8)+c_3(2,1,-1)=(0,0,0)$

$\Rightarrow (c_1+c_2+2c_3, 3c_1-7c_2+c_3, 2c_1-8c_2-c_3) = (0,0,0)$

$\Rightarrow c_1+c_2+2c_3 = 0,\ 3c_1-7c_2+c_3 = 0\ \ 2c_1-8c_2-c_3 = 0$

$\Rightarrow c_1= 3k, c_2 = k, c_3 = -2k$ any arbitrary k.

$\therefore$ $c_1\alpha+c_2\beta+c_3\gamma=0$ need not imply $c_1 = 0, c_2 = 0, c_3 = 0$

$\therefore$ $S = \{\alpha, \beta, \gamma\}$ is linearly dependent.

1.07 Standard properties of linearly independent and dependent sets

Theorem 1 : Let V be a vector space over a field F . then

(i) The set of vectors V containing the null vector is linearly dependent

(ii) The set consisting of single vector α of V is linearly independent if and only if $\alpha \neq 0$

(iii) Every non-empty subset of a linearly independent set of a vectors of V is linearly independent.

(iv) Any super set of a linearly dependent set is linearly dependent.

***Proof* :**

(i) Let S = { $\alpha_{1,}$ $\alpha_{2,...}$ α_n } be a set of vectors of V

containing the zero vector . Let $\alpha_1 = 0$

then 1. α_1 + 0. α_2 + ……..+ 0. $\alpha_n = 0$ $\because$ 1. $\alpha_1 = \alpha_1 = 0$.

$\therefore$ there exists a linear combination of the form

$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots c_n\alpha_n = 0$ in which $c_1 \neq 0$

(i. e, not all $c_i = 0$)

$\therefore$ S is linearly dependent.

(ii) Let { α } be a set consisting of a single vector .

Let { α } be linearly independent . we shall prove that $\alpha \neq 0$

If $\alpha = 0$, then { α } is a set consisting of the null vectors and hence from (i) { α } is linearly dependent .

Which contradicts that { α } is linearly independent.

$\therefore \quad \alpha \neq 0$.

Conversely, let $\alpha \neq 0$ then

c . $\alpha = 0 \Rightarrow c = 0$ or $\alpha = 0$.

But $\alpha \neq 0$ $\therefore$ c = 0

$\therefore$ { α } is linearly independent .

(iii) Let S be a linearly independent subset of V.

Let T be sub set of S

If T is a null set ,then it is linearly independent.

If T is a non-empty set, then S may be a finite set or an infinite set.

(a) Let S be a finite subset of V.

Let S = { $\alpha_{1,}$ $\alpha_{2,}$................ α_n }

Let T = { $\alpha_{1,}$ $\alpha_{2,}$................ α_m} where $1 \leq m \leq n$.

Let $c_{1,}c_{2,}$..................$c_m \in$ F be such that

$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots c_m\alpha_m = 0$

Then $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots c_m\alpha_m + 0\ \alpha_{m+1} + \ldots + 0\ \alpha_n = 0$

$\Rightarrow c_1 = c_2 = \ldots\ldots\ldots\ldots = c_n = 0.$ since S, is linearly independent .

$\therefore$ T is linearly independent set.

(b) Let S be a finite set.

If T is a finite subset of S, then as T is a finite subset of an infinite linearly independent set S,

$\therefore$ T is linearly independent .

If T is an infinite subset of S, let W be any finite subset of T.

$\therefore$ W is linearly independent as Sis linearly independent

$\therefore$ T is linearly independent .

(iv) Let S = $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ be a linearly dependent set so that $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots c_m\alpha_m = 0 \Rightarrow$ at least one $c_1 \neq 0$.

Consider the super set $\{\alpha_1, \alpha_2, \ldots\ldots \alpha_i \ldots \alpha_m, \alpha_n\}$ then

$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_i\alpha_i + \ldots\ldots + c_m\alpha_m + c\alpha_n = 0$

In this equation, there is atleast one $C_i \neq 0$.

Hence the super set $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_1 \ldots\ldots \alpha_m, \alpha\}$ is not linearly independent

$\therefore$ it is linearly dependent .

Theorem 2 : A set of non-zero vectors $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ of a vectors space V(F) is linearly dependent if and only if some one of those vectors say α_k($2 \leq k \leq n$) is expressed as a linear combination of its preceding ones.

***Proof* : (i) condition necessary:**

i.e , if = $\{\alpha_1, \alpha_2, \ldots\ldots \alpha_n\}$is linearly dependent then to prove that α_k is expressed as a linear combination of $\alpha_1, \alpha_2, \ldots\ldots \alpha_{k-1}$ where $2 \leq k \leq$n. Since S is linearly dependent, there exist scalar C_1 not all zero such that $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_k\alpha_k + \ldots\ldots\ldots c_n\alpha_n = 0$

Let C_k be the last non-zero scalar.

If k = 1, then $C_1\alpha_1 = 0 \Rightarrow \alpha_1 = 0 \because c_i \neq 0$.

$\alpha_1 = 0$ is a contradiction to the hypothesis that $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n$ are non-zero vectors .

$\therefore$ $k \neq 1 \therefore 2 \leq k \leq$n.
Now $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_k\alpha_k = 0$ ($c_k \neq 0$)

$$\therefore \alpha_k = \frac{-c_1}{c_k}\alpha_1 - \frac{c_2}{c_k}\alpha_k \ldots\ldots\ldots\ldots \frac{c_{k-1}}{c_k}\alpha_{k-1}$$

$\therefore$ α_k is a linear combination of its preceding ones.

(ii) *Condition sufficient* :

i.e ,if one of the vectors say α_k is expressed as a linear combination of its preceding ones , then set
S = $\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ is linearly dependent .
$\alpha_k = c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_{k-1}\alpha_{k-1}$
$\therefore$ $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_{k-1}\alpha_{k-1} - \alpha_k = 0$.

This is be written as
$c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots\ldots + c_{k-1}\alpha_{k-1} + (-1)\alpha_k + 0\alpha_{k+1}$
$+ \ldots\ldots\ldots + 0\alpha_n = 0.$

In this equation ,there is atleast one scalar –1 which is not Equal to 0 Hence the set S =$\{\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n\}$ is linearly dependent .

***Note* :** Two vectors are linearly dependent iff one is a multiple of the other.

Theorem 3 : A subset S = { (x_1, x_2, x_3), (y_1, y_2, y_3) ,(z_1, z_2, z_3)} of V_3(R) is linearly dependent iff.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

Proof : The set S is linearly dependent if there exist scalars c_1, c_2, c_3 not all zero such that

$c_1(x_1, x_2, x_3) + c_2(y_1, y_2, y_3) + c_3(z_1, z_2, z_3) = (0,0,0)$

i.e , iff $(c_1x_1+ c_2y_1+ c_3z_1,\ c_1x_2+ c_2y_2+ c_3z_2,\ c_1x_3+ c_2y_3+ c_3z_3) = (0,0,0)$
i.e iff the equations

$$c_1x_1+ c_2y_1+ c_3z_1 = 0$$

$$c_1x_2+ c_2y_2+ c_3z_2 = 0$$

$c_1x_3+ c_2y_3+ c_3z_3 = 0$ has a non –trivial solution.

i.e., iff the coefficient matrix $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}^{\prime}$ is singular.

i.e., iff the determinant $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_1 & y_2 & z_3 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

or $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$ since $|A| = |A^T|$.

Hence the proof of the theorem.

i.e The set S is linearly independent if the determinant $\neq 0$

***Note* :** This theorem can be extended to a subset of n vectors in the vector space $V_n(R)$.

Theorem 4 : A set of vectors is linearly dependent if and only if it contains a proper subset spanning the same subspace.

***Proof* :** Let S = { $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n$} be a linearly dependent set of vectors of the vector space V.

Let W be the subspace spanned by the elements of S.

Since S is linearly dependent , it must contain a vector say α_k which is a either 0 or it is expressed as a linear combination of its preceding ones.

Even if we delete this vector α_k from S , still it spans the subspace W.

Repeating this process of deleteing a vector, we arrive at a subset S_1 of S which spans the subspace W and no elements of which is linear combination of its preceding ones.

$\therefore$ the final set S_1 is linearly independent .

Conversely, if S has proper subset S_1, whose elements span the same subspace as S does, then it contains an element which is a linear combination of the elements of itself.

Hence the theorem .

Theorem 5 : A finite set of vector of a vector space V containing non –zero vectors has a linear independent subset which span the same subspace.

Proof : The proof of this theorem is the same as that of the first part of the previous theorem.

Theorem 6 : If n vector span a vector space V, over a field F and r vectors of V are linearly independent then $n \geq r$.

***Proof* :** Let S = { $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_n$} be a set of n vector of V which spans V.

Let T ={$\beta_1, \beta_2, \ldots\ldots\ldots \beta_r$} be a set of r linearly independent vectors of V.

Since S spans V, every vectors of V is a linear combination of $\alpha_{1}, \alpha_{2}, \ldots\ldots \alpha_{n}$.

$\therefore \beta_1 \in$ V is also a linear combination of $\alpha_{1}, \alpha_{2}, \ldots\ldots \alpha_{n}$.

Hence (by theorem 4) the set $T_1=\{\beta_{1}, \alpha_{1}, \alpha_{2}, \ldots\ldots \alpha_{n}\}$ is linearly dependent and spans V.

$\therefore$ There exists a vector say $\alpha_i \in T_I$ which is a linear combination of the proceeding ones.

This cannot be β_1 since it belongs to the linearly independent T.

Deleting this vector α_I from T_I we get

$S_1=\{\beta_1, \alpha_1, \alpha_2, \ldots\ldots \alpha_{i-1}, \alpha_{i+1}, \ldots\ldots \alpha_n\}$

Now S_1 is linearly dependent and still spans V.

$\therefore \beta_2 \in$ V is a linear combination of the elements of S_1

Hence the set $T_\alpha=\{\beta_2, \beta_1, \alpha_1, \alpha_2, \ldots\ldots \alpha_n\}$ is linearly dependent and spans V.

$\therefore$ there exists a vector say $\alpha_j \in T_\alpha$ which is a linear combination of the preceding ones.

This cannot be β_α since it belongs to the linearly independent set T.

Deleting this vector α_j from T_α we get

$S_\alpha = \{\beta_2, \beta_1, \alpha_1, \alpha_2, \ldots\ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_{j-1} \alpha_{j+1} \ldots\ldots \alpha_n\}$

Which still generate V.

Repeat this process of deleting one α and including one β, till all the β's are exhausted.

To do this , the number of α's must be greater than or equal to the number of β's. i.e ,n $\geq$ r. which proves the theorem.

Worked Problems

(1) Prove that the set S = { (1,2,1),(2,1,0),(1,-1,2) } is linearly independent.

Solution : Consider the determinant

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 1(2 - 0) - 2(4 - 0) + 1(-2 - 1)$$

$$= 2 - 8 - 3 = -9 \neq 0.$$

$\therefore$ The set S is linearly independent.

(2) Show that the vectors (1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0) and (2, 1, 1, 6) are linearly dependent in R^4.

Solution : Consider $\begin{vmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{vmatrix}$

$C_2 - C_1, C_3 + (-2)C_1, C_4 + (-4) C_1$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -3 & -9 & -6 \\ 1 & -2 & -6 & -4 \\ 2 & -1 & -3 & -2 \end{vmatrix} = 1 \begin{vmatrix} -3 & -9 & -6 \\ -2 & -6 & -4 \\ -1 & -3 & -2 \end{vmatrix}$$

$$= (-3)(-2)(-1) \begin{vmatrix} 1 & 3 & 2 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \end{vmatrix} = 0$$

$\therefore$ Given set is L.D.

(3) Show that S = { (1,2,4), 1,0,0), (0, 1, 0) 0,0,1) } is linearly dependent in V_3 (R).

***Solution* :** S can be written as

S = { (1,0,0), (0,1,0) (0,0,1) ,(1,2,4).}

Consider 1 (1,0,0) + 2 (0,1,0) + 4 (0,0,1)

= (1,0,0) + (0,2,0) + (0,0,4)

= (1,2,4)

$\therefore$ (1,2,4) can be expressed as a linear combination of its preceding vectors as 1 (1,0,0) + 2 (0,1,0) + 4 (0,0,1)

$\therefore$ S is linearly dependent .

(4) Find whether the set = { x^2– 1, x + 1, x -1} is linearly independent in the vector space of all polynomials over the field of real numbers.

***Solution* :** S = { x^2 –1,x+1,x –1}

Consider $c_1(x^2 -1) + c_2 (x+1) + c_3(x -1) = 0.$

$\Rightarrow \quad c_2 x^2 - c_1 + c_2 x + c_2 + c_3 x - c_3 = 0$

$\Rightarrow \quad c_1 x^2 + (c_2+c_3) x + (-c_1 + c_2 - c_3) = 0$

$\Rightarrow \quad c_1 = 0 , c_2 + c_3 = 0, -c_1 + c_2 - c_3 = 0$

$\Rightarrow \quad c_2 + c_3 = 0, c_2 - c_3 = 0 \Rightarrow c_2 = 0, c_3 = 0$

$\therefore \quad c_1 (x^2-1) + c_2 (x-1) + c_3(x-1) = 0 \Rightarrow c_1=0, c_2= 0, c_3=0$

$\therefore$ S is linearly independent .

(5) Prove that the four vector α_1 = (1,0,0), α_2= (0,1,0), α_3= (0,0,1), α_4= (1,1,1) in V_3 (R) are linearly dependent but any three of them are linearly independent.

***Solution* :**

Let $c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) + c_4 (1,1,1) = (0,0,0)$

$\Rightarrow (c_1 + c_4, c_2 + c_4, c_3 + c_4) = (0,0,0)$

$\Rightarrow c_1 + c_4 = 0 \; c_2 + c_4 = 0 \; c_3 + c_4 = 0.$

$\therefore$ If $c_4 = -k, c_1 = k, c_2 = k \; c_3 = k$. Now choosing k = 1, we get

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = 0 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0.$

$\therefore \{ \alpha_1 , \alpha_2 , \alpha_3 , \alpha_4\}$ is L.D.

Now let us show that $\{\alpha_1 \; \alpha_2 \; \alpha_3\}$ is L.I.

$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 0$

$\Rightarrow c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$\Rightarrow (c_1, c_2, c_3) = (0,0,0)$

$\Rightarrow c_1 = 0, \; c_2 = 0, \; c_3 = 0$

$\therefore \{\alpha_1, \; \alpha_2, \alpha_3\}$ is L.I

(6) If α, β, γ are linearly independent vectors in a vector space V(F), then prove that

(i) $\alpha + \beta, \beta + \gamma, \gamma + \alpha$

(ii) $\alpha + \beta, \alpha - \beta, \alpha - 2\beta + \gamma$ are also linearly independent.

***Solution* :** (i) Consider

$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$

$\Rightarrow a\alpha + a\beta + b\beta + b\gamma + c\gamma + c\alpha = 0$

$\Rightarrow (a\alpha + c\alpha) + (a\beta + b\beta) + (b\gamma + c\gamma) = 0$

$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0$

$\Rightarrow (a + c) = 0, (a + b) = 0 \; (b + c) = 0$ since α, β, γ are L.I.

$\Rightarrow a = 0, b = 0, c = 0$ (by solving the equations)

$\Rightarrow \alpha + \beta, \; \beta + \gamma, \; \gamma + \alpha$ are linearly independent.

(ii) $a(\alpha + \beta) + b(\alpha - \beta) + c(\alpha - 2\beta + \gamma) = 0$

$\Rightarrow a\alpha + a\beta + b\alpha - b\beta + c\alpha - 2c\beta + c\gamma = 0$

$\Rightarrow a\alpha + c\alpha + a\beta - b\beta - 2c\beta + c\gamma = 0$

$\Rightarrow (a + c)\alpha + (a - b - 2c)\beta + c\gamma = 0$

$\Rightarrow a + c = 0 \; a - b - 2c = 0 \; c = 0$ since $\alpha, \beta \; \gamma$ are L.I

$\Rightarrow a = 0, b = 0, c = 0$ (by solving the equations.)

$\Rightarrow \alpha + \beta, \alpha - \beta, \alpha - 2\beta + \gamma$ are linearly independent.

EXERCISE

1. Show that S = {(1, 2, 3) (1, 0, 0) (0, 1, 0) (0, 0,1)} is a linear dependent subset of the vector space R^3 (R).

2. Show that $\{1, x, x^2, x^3, \ldots\}$ is Linear independent of vector space F(x) of all polynomials over the field F.

3. Prove that if two vectors are L.I one of them is a scalar multiple of the other.

4. Prove that the set of vectors which contains the zero vector is L.D.

5. Prove that every superset of linear dependent set of vectors is L.D.

6. Show that the following vectors in $V_3(R)$ are L.D
   a) (1, 2, 3) (4, 1, 5) (-4, 6, 2)
   b) (3, 0,-3) (-1, 1, 2) (4, 2, -2) (2, 1, 1)
   c) (1, 1, 2, 4) (2, -1, -5, 2) (1, -1, -4, 0) (2, 1, 1, 6)

7. Show that the following vectors in V_3 (R) are L. I
   a) {(1, 1, 1) (1, 0, 0) (0, 1, 0) (0, 0, 1)}
   b) {(1, 2, -3) (1, -3, 2), (2, -1, 5)}
   c) {(1, 1, -1) (2, -3, 5) (-2, 1, 4)}
   d) {(1, 1, 1) (2, 2, 0) (3, 0, 0)}

8. Which of the following set of vector are L.D
   a) {(2, -1, 4) (3, 6, 2) (2, 10, -4)}
   b) {(1, 1, 1) (2, 2, 0) (3, 0, 0)}
   c) {(1, 3, 3) (0, 1, 4) (5, 6, 3) (7, 2, -1)}
   d) {(1, 2, 1, 2) (3, 2, 3,2) (-1, -3, 0, 4) (0, 4, -1, -3)}
   e) {(1, 0, 0, 0) (1, 1, 0, 0) (1, 1, 1, 1) (0, 0, 1, 1)}
   f) {(1, 0, 1), (1, 1, 0) (-1, 0, -1)}
   g) {(1, 2, 1) (-1, 1, 0) (5, -1, 2)}

9. Which of the following subset S of the vector space of all real valued functions defined over the interval (0, ∞) linear independent (L. I)

   1. S = {x, sinx cosx} 2. $\{x, x^2, e^{2x}\}$
   3. S = {cosx, sinx, sin(x +1)} 4. {logx, 2logx, 3logx}
   5. $S = \{\cos 2x, \cos^2 x, \sin^2 x\}$ 6. {1, sinx, sin2x}

10. Which of the following subsets of the vector space of all real valued functions defined over the interval (0, ∞) Linearly dependent (L.D)
   a) $\{x^2 - 4, x + 2, x - \frac{2}{3}x^2\}$
   b) $\{2 - x + 4x^2, 2 + 10x - 4x^2, 3 + 6x + 2x^2\}$
   c) $\{1 + 3x + 3x^2, x + 4x^2, 5 + 6x + 3x^2, 7 + 2x - x^2\}$
   d) $\{3 + x + x^2, 2 - x + 5x^2, 4 - 3x^2\}$
   c) $\{2x^3 + x^2 + x + 1, x^3 + 3x^2 + x - 2, x^3 + 2x^2 - x + 3\}$

Answers

1. (1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) – (2, -1, 1)
3. $(1, 7 -4) = -3\alpha_1 + 2\alpha_2$
4. No. 6. a) K = -8 b) K = -8

8. a, b, e, g – independent ; c, d, f – L. D

9. (1) (2), - L. I 10) a c) - L.D..

1.08 : Basis and Dimension

***Definition* :** Let V be a vector space over a field F . A subset B of V is called a basis of V if (i) B is linearly independent and

(ii) B spans V.

***Definition* :** A vector space V is said tobe finite dimensional if it has a finite basis.

Definition : The *dimension* of a finite dimensional vector space V over a field F is the number of elements in a basis of V and is denoted by dim V.

Worked Example :

(1) The set S = { (1,0,0),(0,10),(0,0,1)} is a basis of V_3(R).

***Solution* :** (i) S is linearly independent.

Consider $c_1 (1, 0, 0) + c_2(0,10) + c_3(0, 0, 1) = (0, 0, 0)$

$\Rightarrow (c_1, c_2, c_3) = (0, 0, 0)$

$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$

$\therefore c_1 (1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1) = 0$

$\Rightarrow C_1 = 0, C_2 = 0, C_3 = 0$

$\therefore$ S is linearly independent.

(ii) S spans V_3(R). ie, any vector (x_1, x_2, x_3) in V_3(R). can be expressed as a linear combination of (1, 0, 0), (0, 1, 0), (0, 0, 1) .

$$(x_1, x_2, x_3) = x_1 (1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0,1)$$

Hence S is a basis of $V_3(R)$.

These vectors are denoted by e_1, e_2, e_3 repectively and are called *standard basis vectors* and S is called the standard basis.

Since the basis contains 3 elements ,the vector space is finite dimensional and Dim V = 3

(2) Determine whether the set S = {(2,1),(1,-2),(1,0)} is a basis of R^2. (M O2)

***Solution* :** Consider $c_1(2,1)+c_2(1,-2)+c_3(1,0) = (0,0)$

$\Rightarrow \quad (2c_1+ c_2+ c_3, c_1-2c_2) = (0,0)$

$\Rightarrow \quad 2c_1+ c_2+ c_3 = 0 \; c_1-2c_2 = 0$

$\Rightarrow \quad 2c_1+ c_2 = - c_3 \; c_1-2c_2 = 0$

$\Rightarrow c_1 = \frac{-2c_3}{5}, c_2 = \frac{-c_3}{5}$ *and* c_3 is arbitrary.

$\therefore$ S is not linearly independent .

Hence S is not a basis of R^2.

(3) Show that the infinite set S = {1, x,x^2,………..x^n ,……} is a basis of the vector space F [x] of all polynomials over the field F.

***Solution* :** (i) In order to show that S is LI:, we have to show that every finite subset of S is L.I.

Let T = { $x^{m1}, x^{m2}, \ldots\ldots x^{mr}$ } be an arbitrary finite subset of S, so that each m_1 is a non – negative integer.

For any scalars $a_1, a_2, \ldots\ldots a_r$ we have

$a_1x^{m1}+a_2x^{m2}+\ldots\ldots+a_rx^{mr} = 0$

$\Rightarrow a_1= 0, a_2 = 0,\ldots\ldots a_r = 0$

since by equality of polynomials, the coefficients are 0.

This proves that S is L.I.

If f(x) is an arbitrary member of F [x] of all polynomials,then we can write

$f(x) = a_0.1+ a_1x + a_2x^2 + \ldots\ldots\ldots+ a_mx^m$.

i.e., f(x) can be expressed as a linear conbination of a finite number of elements of S.

$\therefore$ S spans F[x] .

$\therefore$ S is a basis of F[x].

***Note* :** (a) The vector space F[x] has no finite basis .

(b) F[x] is an infinite dimensional vector space.

Theorem on basis and dimension

Theorem 1: Any two bases of a finite dimensional vector space V have the same finite number of elements.

***Proof* :** Let V be a finite dimensional vector space over a field F.

Let S={$\alpha_1, \alpha_2,\ldots\ldots \alpha_n$}and T = {$\beta_1, \beta_2,\ldots\ldots \beta_n$} be two bases of V.

We have to prove that n = m.

Since S is a basis ,S spans V.

Since T is a basis ,T is linearly independent.

$\therefore$ (by Theorem 6 on linearly independent vectors of section 2.07) $m \leq n$ ……………..(1)

Similarly ,since S is a basis ,S is L.I

and since T is a basis ,T spans V.

$\therefore$ (by the same theorem 6) $n \leq m$ ………(2)

from (1) and (2), m= n.

Theorem 2 : **In an n-dimensional vector space V,**

(i) any (n + 1) elements of V are linearly dependent.

(ii) none of the set of (n- 1) elements can span V

Proof **:** (i) Let S = { α_1, α_2,………… α_n} be a basis of an n dimensional vector space V.

Let T be any set consisting of (n+1) elements .

Since S is a basis ,it spans V.

If T is linearly independent , then we must have the number of elements in T less than or equal to the number of elements in S.

But T has more elements i.e,(n +1) than in S (i.e,n)

$\therefore$ T is linearly dependent .

$\therefore$ any (n + 1) elements of V are linearly dependent.

(ii) Let S = { α_1, α_2,………… α_n} be a basis of V.

Let T be any set containing (n- 1) elements.

Since S is a basis ,it is linearly independent.

If T generates the entire space V, then the set T must contain more or equal number of vectors than in S. But has (n-1) which is < n elements .

Hence T cannot span V.

$\therefore$ No set of (n-1) elements can span V.

Theorem 3 : Any linearly independent set of elements of a finite dimensional vector space V is a part of a basis.

Proof **:** Let S= {α_1, α_2,………… α_k} be a linearly independent subset of an n dimensional vector space V.

Now we shall determine vectors α_{k+1}, α_{k+2},……… α_n

Such that { α_1, α_2,………α_k, α_{k+1}, ,…… α_n} is a basis of V.

Clearly $k \leq n$.

If k = n ,then clearly S is a basis of V, since any linearly independent subset of V is a basis of V.

If k < n , then S is not a basis of V.

Let T be the subspace spanned by the vectors of S.

Since S is linearly independent, we have $T \neq V$.

i.e, T is a proper subset of V.

$\therefore$ there exists a non zero vector $\alpha_{k+1} \in V$ such that $\alpha_{k+1} \notin T$.

$\therefore$ The set {α_1, α_2,………… α_k, α_{k+1}} is linearly independent.

If k +1 = n then { α_1, α_2,………… α_k, α_{k+1}} is a basis of V.

If $k+1 \neq n$, we repeat the above process till we get n linearly independent vectors $\alpha_1, \alpha_2, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n$ which form a basis of V.

Theorem 4: For n vector of n dimensional vector space V to be a basis, it is sufficient that they span V or that they are linearly independent.

Proof **:** Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ span V.

$\therefore$ there exists a linearly independent subset T of S which also spans V

$\therefore$ T is a basis .

Since dim V = n ,the number of elements in T is n.

But T is a subset of S which also has n elements.

$\therefore$ T = S and hence S is a basis of V.

Secondly if S is linearly independent, then it is a part of a basis (by theorem 3) and this basis has n elements ($\because$ dim V = n) and hence S itself is a basis.

Theorem 5 : Let A be any $m \times n$ matrix which is equivalent to a row reduced echelon matrix E. Then the non-zero rows of E form a basis of the subspace spanned by the rows of A.

Proof **:** Since E is the echelon form of A, it follows that the non-zero rows of E are linearly independent, and hence form a basis of the subspace spanned by the rows of E.

Since A and E are equivalent, the rows of A and E generate the same subspace.

$\therefore$ the non-zero rows of E form a basis of the subspace spanned by the rows of A.

Note **:** (i) Since the dimension of a vector space is the number of elements in a basis, the number of non –zero rows in E is the dimension of the subspace spanned by the rows of A.

(ii) Since the rank of a matrix is the number of non-zero rows, the dimension of the subspace spanned by the rows of A is equal to the rank of A.

(iii) To find the basis and the dimension of a subspace spanned the vectors, reduce the matrix whose rows are the given vectors to echelon form.

Worked Examples :

1. Determine whether the set of vectors

{(1, 2, 3) (-2, 1, 3), (3, 1,0) } is a basis of R^3. **(N 2002)**

Solution : Consider $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \\ 3 & 1 & 0 \end{vmatrix}$

$= 1(0 - 3) -2 (0 - 9) + 3(-2 -3$

$= -3 + 18 - 15 = 0$

$\therefore$ The vectors are L.D

$\therefore$ It is not a basis of R^3

2. Define basis and dimension of a vector space. Determine the basis of the subspace spanned by the vectors

$$\left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}, \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix} \right\}$$

Solution **:** Let S be the above set. S = {A, B, C, D}

$$\begin{bmatrix} 1 & -5 & -4 & 2 \\ 1 & 1 & -1 & 5 \\ 2 & -4 & -5 & 7 \\ 1 & -7 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 0 & -2 & -1 & -1 \end{bmatrix} \begin{matrix} R_3 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -5 & -4 & 2 \\ 0 & 6 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 - R_1 \\ R_4 + \frac{1}{3}R_2 \end{matrix}$$

The final matrix has two non-zero rows

$\therefore$ Subspace is $\left\{ \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 6 \\ 3 & 3 \end{bmatrix} \right\}$

$\therefore$ Dimension of the subspace = 2

3. Find the basis and dimension of the subspace spanned by the vectors (1, 2, 0), (1, 1, 1) (2, 0, 1) of the vector space $V_3(z_3)$ where z_3 is the field of integer modulo 3.

***Solution* :** Let S = {(1, 2, 0), (1, 1,1) , (2, 0, 1)}

Consider A = $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -4 & 1 \end{bmatrix}$

$|A|$ = 1(1-0) – 2(1 - 2) + 0

= 1 + 2 = 0 under + mod 3.

$\therefore$ S is linearly dependent set.

To find the subspace of A

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_2 +_3 2R_1 \\ R_3 +_3 R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 + 2R_1$$

In the final matrix has two non zero rows.

$\therefore$ subspace is $\{(1,-2,3)(0,-1,1)\}$

Dimension of the subspace = 2

$\therefore$ S is linearly dependent set.

To find the subspace of A

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_2 \ +_3 \ 2R_1 \\ R_3 \ +_3 \ R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \ + \ 2R_1$$

In the final matrix has two non zero rows. Thus the subspace

Is $\{(1,2,0),(0,2,1)\}$ and its dim = 2.

4. Find the dimension and basis of the subspace spanned by the vectors {(2, 4, 2) (1 , -1, 0) (1, 2, 1), (0, 3, 1)} in V_3(R)

(M 02, M 2000)

***Solution* :** Let S be the given set. D[V_3(R)] = 3

Any subset of V_3(R) containing more than 3 vector are L.D.

Consider $\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \quad \frac{1}{2}R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad R_2 - R_1 \quad R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_4 + R_2$$

In the last matrix has two non zero rows

$\therefore$ Subspace is S^1 = {(1, 2, 1) (0, -3, -1)}

The dimension of the subspace S^1 ie $d(S^1) = 2$.

5. Define basis and dimension of a vector space. Find basis and dimension of subspace of V_3 (R) spanned by

{(1 , -2, 3) (1, -3, 4) (-1, 1, -2)} (N 02) (M 01)

Solution : Consider A = $\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix}$

$|A|$ = 1(6 – 4) + 2 (-2 + 4) + 3 (1 – 3)

= 2 + 4 – 6 = 0

$\therefore$ Given set is L.D

$\therefore$ It is not a basis of V_3(R).

To find the dimension and basis of the subspace of S

Consider A = $\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$

$$\sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 - R_2$$

The final matrix has two non-zero rows

$\therefore$ Subspace is {(1, 2,, 3) (0, -1, 1)}

Dimension of the subspace = 2.

6. Prove that $\{(1,2,1),(3,4,-7)(3,1,5)\}$ is a basis of $V_3(R)$

***Solution*: Let** $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & -7 \\ 3 & 1 & 5 \end{bmatrix}$ 〈 N 2001 〉

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -10 \\ 0 & -5 & 2 \end{bmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$|A| = 1(-4-50) \neq 0.$$

$\therefore$ It is L.I. $\therefore$ It is a basis of $V_3(R)$

7. Define basis and dimension of a vector space. Examine whether the set of vectors $(2,1,0),(1,1,2)$ *and* $(1,2,1)$ is a basis of the space $V_3(R)$.

***Solution*:** Define basis and dimension.

Consider $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$|A| = 1(1-0)+1(2-0)+2(4-1)$$

$$= 1+2+6 \neq 0$$

$\therefore$ Given set of vectors are L. I.

It is a basis of $V_3(R)$

(8) Show that the vectors (1,0,-1), (1,2,1), (0,-3,2) form a basis of V_3(R)

***Solution*:** Let α = (1,0,-1) β = (1,2,1) γ = (0,-3,2)

Consider $\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{vmatrix}$ = 1(4 + 3) –1, (-3, -0) = 7 + 3 = 10 $\neq$ 0

$\therefore$ The set $\{\alpha,\beta,\gamma\}$ is L.I

Any vector (x_1,x_2,x_3) in V_3(R) can be expressed as a linear combination of α,β,γ .

Let (x_1, x_2, x_3) = a = (1,0,-1) +b (1,2,1) + c (0,-3,2)

= (a + b ,2b – 3c, -a + b +2c)

$\therefore$ a + b = x_1 , 2b – 3c = x_2 -a + b +2c = x_3.

Adding first and third equations,we get 2b+2c = x_1+ x_3.

Now 2b+2c = x_1+ x_3

2b – 3c = x_2

subtracting 5c = x_1+ x_3- x_2

$$\therefore \quad c = \frac{x_1+x_3-x_2}{5}$$

2b – 3c = x_2

$$\therefore 2b - \frac{3(x_1+x_3-x_2)}{5} = x_2$$

or 10b = 5x_2 + 3(x_1+ x_3- x_2)

= 2x_2 + 3x_1+ 3x_3.

$$\therefore b = \frac{2x_2+3x_1+3x_3}{10}$$

Now a + b = x_1

i.e, a = x_1 - $\dfrac{2X_2+3x_1+3x_3}{10}$

$$= \frac{10x_1-2x_2-3x_1-3x_3}{10}$$

$$a = \frac{7x_1-2x_2-3x}{10}$$

Hence (x_1, x_2, x_3) can be expressed as a linear cobination . Hence $\{\alpha, \beta, \gamma\}$ spans V_3R. $\therefore$ $\{\alpha, \beta, \gamma\}$ is a basis of V_3(R).

9. Determine a basis of a subspace spanned by the vectors

(2,-3, 1),(3,0,1),(0,2,1), (1,1,1) of V_3 R.

Solution : Let S = { (2,-3, 1),(3,0,1),(0,2,1), (1,1,1) }

S contain 4 elements and dim V_3 R. = 3

$\therefore$ S is linearly dependent.

Now consider the matrix of vectors

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Let us reduce A to echelon form using elementary row transformations.

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & -3 & 1 \end{bmatrix} (R_1 \leftrightarrow R_4)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 2 & 1 \\ 2 & -5 & -1 \end{bmatrix} \quad (R_2 - 3R_1) \text{ and } (R_4 - 2R_1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 6 & 3 \\ 0 & -15 & -3 \end{bmatrix} 3(R_2) \text{ and } 3(R_4)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 7 \end{bmatrix} (R_3 + 2R_2) \text{ and } (R_4 - 5R_2)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_4 + 7R_2.$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left(-\frac{1}{3}\right) R_2 \text{ and } (-1) R_3.$$

This is in the echelon form . There are 3 non-zero rows.

$\therefore$ Corresponding to these nonzero rows the vector are (1,1,1), (3,0,1), (0,2,1) and these form a basis of the subspace spanned by S.

$\therefore$ Basis is (1,1,1) ,(3,0,1),(0,2,1) and basis has 3 elements and hence dimension of the subspace is 3.

10. Show that the vectors (1.i.0),(2i,1,1),(0,1+ I,1-i) form a basis of V_3(c)

***Solution* :** Let S= { (1, i ,0) ,(2i,1,1),(0,1+i, 1-i) }

Consider the matrix of the vectors

$$A=\begin{bmatrix}1 & i & 0\\ 2i & 1 & 1\\ 0 & 1+i & 1-i\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & i & 0\\ 0 & 3 & 1\\ 0 & 1+i & 1-i\end{bmatrix} R_2-2i(R_1)$$

$$\sim\begin{bmatrix}1 & i & 0\\ 0 & 3 & 1\\ 0 & 3(1+i) & 3(1-i)\end{bmatrix} 3(R_3)$$

$$\sim\begin{bmatrix}1 & i & 0\\ 0 & 3 & 1\\ 0 & 0 & 2-4i\end{bmatrix} R_3-(1+i)R_2$$

$$\sim\begin{bmatrix}1 & i & 0\\ 0 & 1 & \frac{1}{3}\\ 0 & 0 & 1\end{bmatrix} \frac{1}{3}(R_2) \quad \text{and} \quad \frac{1}{2-4i}(R_3)$$

This is in the echelon form . There are three non-zero rows.

$\therefore$ These non-zero rows determine the basis.

Corresponding to these, the vectors are (1, i, 0), (2i,1,1), (0,1+i, 1-i)}

These form the basis of V_3(c) and hence the dimension of V_3(c) is 3.

11. In a vector space V_3(R), let α = (1, 2, 1) , β = (3, 1,5), γ = (-1, 3, -3). Prove that the subspace spanned by {α β} and { α , β , γ} are the same .

Solution : Consider the matrix

$$A=\begin{bmatrix}1 & 2 & 1\\ 3 & 1 & 5\\ -1 & 3 & -3\end{bmatrix}$$

$$\text{Det } A = \begin{vmatrix}1 & 2 & 1\\ 3 & 1 & 5\\ -1 & 3 & -3\end{vmatrix} = 1(-3-15) - 2(-9+5) + 1(9+1)$$

$$= -18 + 8 + 10 = 0$$

$\therefore$ The set S = $\{\alpha, \beta, \gamma\}$ is L.D

$\therefore$ It has a subset which spans the same subspace as the given set of vectors.

$$\text{Now } A \sim \begin{bmatrix}1 & 2 & 1\\ 0 & -5 & 2\\ 0 & 5 & -2\end{bmatrix} (R_2-3R_1) \text{ and } (R_3+R_1)$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} (R_3 + R_2) \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 \end{bmatrix} \left(\frac{-1}{5}\right)(R_2)$$

This is in the echelon form .There are two non-zero rows.

Corresponding to these non-zero rows,the vectors are

$\alpha = (1,2,1)$ and $\beta = (3,1,5)$.

$\therefore \{\alpha\ \beta\}$ and $\{\alpha, \beta, \gamma\}$ span the same subspace.

12. Show that S = $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ **form a basis of the vector space M₂® of 2 x 2 matrices. Find its dimension.**

Solution : Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ S spans M_2®

and $C_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + C_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + C_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + C_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0.$

$\therefore$ S is L. I

Hence S is a basis of M_2 (R)

Since S contains 4 elements, dim [M_2 (R)] = 4

13. Find the dimension and basis of the subspace spanned by {(1, 3, 2, 4) , (1, 5, -2, 4), (1, 2, 3, 4), (1, 6, -3, 4) } in V_4 (R).

Solution : Consider $\begin{vmatrix} 1 & 3 & 2 & 4 \\ 1 & 5 & -2 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 6 & -3 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & -4 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -5 & 0 \end{vmatrix} R_2 - R_1, R_3 - R_1, R_4 - R_1$$

$$= 1\begin{vmatrix} 2 & -4 & 0 \\ -1 & 1 & 0 \\ 3 & -5 & 0 \end{vmatrix} = 0$$

$\therefore$ The vectors are linearly dependent.

$\therefore$ The four vectors do not form a basis.

Consider the three vectors {(1, 3, 2, 4), (1, 5, -2, 4), (1, 2, 3, 4)}

The matrix of the vectors

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 5 & -2 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1 & 2 & -4 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} R_2 - R_1 \text{ and } R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \frac{1}{2}(R_2)$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This is in the echelon form and there are three non-zero rows.

$\therefore$ The three vectors corresponding to these three non-zero rows are (1, 3, 2, 4), (1, 5, -2, 4), (1, 2, 3, 4). They form a basis of V_4 (R) and dim [V_4 (R)] = 3

14. Determine the dimension and basis for the solution space of the system x + y + z = 0 ,3x + 2y –2z = 0, 4x + 3y – z = 0, 6x + 5y + z= 0.

Solution : x + y + z = 0 --------(1)

3x + 2y –2z = 0 ----------(2)

4x + 3y – z = 0 -------(3)

6x + 5y + z= 0 --------(4)

(1) $\times$ 3 – (2) $\Rightarrow$ y + 5z = 0 -------------(5)

(1) $\times$ 4 - (3) $\Rightarrow$ y + 5z = 0 --------------(6)

(1) $\times$6 - (4) $\Rightarrow$ y + 5z = 0 ------------(7)

$\therefore$ (5) or (6) or(7) $\Rightarrow$ y = -5z

$\therefore$ (1) $\Rightarrow$ x = 4z

$\therefore$ If z = k,x = 4k, y = - 5 k.

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4k \\ -5k \\ k \end{bmatrix} = k \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

$\therefore$ Basis = { (4, -5,1) }

$\therefore$ Dimension = 1

15. Extend the linearly dependent set { (0,1,2) ,(3,2,1)} to a basis of R^3.

Solution : Let S = { (0,1,2) ,(3,2,1)}

(0,1,2) = a (3,2,1)

= (3a ,2a,a)

$\therefore$ 3a = 0, 2a = 1 , a = 2

These equation are inconsistent

$\therefore$ It is not possible to express (0,1,2) as a (3,2,1)

$\therefore$ S is L.I

Include the vector (1,0,0) to S .

Consider the matrix of the vectors.

$$A=\begin{bmatrix}1&0&0\\0&1&2\\3&2&1\end{bmatrix}$$

$$\sim\begin{bmatrix}1&0&0\\0&1&2\\0&2&1\end{bmatrix}R_3-3R_1$$

$$\sim\begin{bmatrix}1&0&0\\0&1&2\\0&0&-3\end{bmatrix}R_3-2R_2$$

$$\sim\begin{bmatrix}1&0&0\\0&1&2\\0&0&1\end{bmatrix}\left(-\frac{1}{3}\right)R_3$$

This is in the echelon form . There are three non-zero rows in this

$\therefore$ The non-zero rows form a basis , corresponding to these non-zero rows, the vectors are (1,0,0) ,90,1,2) ,(3,2,1)

$\therefore$\{ (1,0,0) ,(0,1,2) ,(3,2,1)\} is a basis of R^3 and its dimension = 3.

EXERCISE

(1) Verify whether the following sets of vectors form bases of V_2(R) or V_3(R). If not, find a basis and the dimension of the subspaces spanned by these vectors.

(i) { (2,1), (3,0),} (ii) { (4,1), (-7,-8) }

(iii) {(0,0), (1,3),} (iv){(3,9), (-4,-12)}

(v) {(2,1), (1,-1), (0, 2)} (vi){(1,2,3), (-2,1,3), (3,1,0)}

(vii) {(3,1,-4), (2,5,6), (1,4,8)},

(viii) {(2,-3,1), (4,1,1), (0,-7,1)}

(ix) {(1,-1.0), (0,31), (1,2,1), (2,4,2)}

(x) {(1,6,4), (2,4,-1), (-1,2,5)},

(xi) { (1,2,2,1),(0,2,0,1)(1,-2,2,-1)}

(xiii) { (1,3,2,4), (1,5,-2.4), (1,2,3,4), (1,6,-3,4)}

(2) Which of the following sets of vectors are bases of the vector space of polynomials :

(i) $\{ 1- 3x + 2x^2, 1 + x +4x^2, 1- 7x\}$ in p_2

(ii) $\{ x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + \frac{1}{2} \}$ in p_4

(iii) $\{ 4 + 6x + x^2 , -1 + 4x + 2x^2 , 5 + 2x- x^2\}$ in P_2

(iv) $\{1 + x + x^2, x + x^2 , x^2\}$ in P_2

(v) $\{-4 + x + 3x^2, 6 + 5x + 2x^2, 8 + 4x + x^2 \}$ in P_2

(3) Determine the dimension and basis of the solution space of the system of equations :

(i) $x + y - z = 0, -2x + y + 2z = 0 , -x + z = 0$

(ii) $2x + y + 3z = 0, x + 5z = 0, y + z = 0$ (M 02)

(iii) $x - 3y + 2z = 0,\ 2x - 6y + 2z = 0,\ 3x - 9y + 3z = 0$

(iv) $x - 4y + 3z - w = 0,\ x - 8y + 6z - 2w = 0.$

(4) Which of the following set of vectors is a basis of the space of all 2 $\times$2 matrices over R.

(i) $\left\{\begin{bmatrix}3 & 6\\ 3 & -6\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & -8\\ -12 & -4\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}\right\}$

(ii) $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}\right\}$

(iii) $\left\{\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix}\begin{bmatrix}-1 & 2\\ -2 & -2\end{bmatrix}\begin{bmatrix}0 & 5\\ 0 & -1\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\right\}$

(iv) $\left\{\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right\}$

(v) $\left\{\begin{pmatrix}1 & -5\\ -4 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 5\end{pmatrix}\begin{pmatrix}2 & -4\\ -5 & 7\end{pmatrix}\begin{pmatrix}1 & -7\\ -5 & 1\end{pmatrix}\right\}$

(5) Let be the space spanned by $\alpha = \cos^2 x$, $\beta = \sin^2 x$, $\gamma = \cos 2x$. Is $\{\alpha, \beta, \gamma\}$ a basis of W ? If not , find a basis and dimension.

Answers

(1) (i) Basis (ii) basis

(iii) not a basis ;Basis { (1,3) } ; dim = 1

(iv) not a basis ;Basis { (1,3) } ; dim = 1

(v) not a basis ;Basis { (2,1),(1-1)}; dim = 2

(vi) Basis

(vii) basis ;Basis {(1,2,3) (3,1,0)} ; dim = 2

(viii) not a basis ;Basis { (2,-3,1), (4,1,1) }; dim =2

(ix) not a basis ;Basis { (2,4,2),(1,-1,0)} ; dim = 2

(x) not a basis ;Basis { (1,6,4),(2,4,-1) }; dim = 2

(xi) not a basis ;Basis {(1,2,2,1),(0,2,0,1)}; dim = 2

(xii) not a basis; Basis {(1,3,2,4),(1,5,-2,4), (1,2,3,4)}; dim = 3

(2) (i) not a basis (ii) basis (iii) not a basis

(iv) basis (v) basis

(3) (i) Basis = {(1,0,1)}; dim = 1

(ii) has no basis ; dim= 0

(iii) Basis = {(3,1,0) ,(-10,1)}; dim = 2

(iv) Basis = {(4,1,0,0), (-3,0,1,0), (1,0,0,1)};dim = 3

(4) (i) Basis, (ii) asis, (iii) not a basis,

(iv) Basis (v) not a basis.

(5) not a basis ; any two of α, β, γ form a basis; dim = 2

1.09 Linear Transformations

In this section ,we study mapping from one vector space into another vector space .

***Definition* :** Let U and V be two vector spaces over a field F. Then the mapping T:U $\rightarrow$V is said to be a linear transformation if

(i) T (α + β) = T (α) + T (β) $\forall \alpha, \beta \in U$

and (ii) T (c.α) = c.T (α) ,$\forall c \in F$ and $\alpha \in U$.

***Definition* :** Let U be vector space over a field F. Then the linear transformation T : U $\rightarrow$ U is called a linear map on u

***Note* :** (1) In the linear transformation , u and v are vector spaces over the same field F.

(2) In the condition the + sign on the LHS is the + of the vector space u and + sign on the RHS is the + of the vector space V. similarly the scalar multiplication on the LHS is the scalar multiplication of u and that on the RHS is the scalar multiplication of v.

Worked Examples :

(1) T: V_2 (R) $\rightarrow$ V_2 (R) is defined` by

T (x,y) = (3x + 2y ,3x –4y).verify whether T is a linear transformation.

***Solution* :** α = (x_1,, y_1) , β = (x_2, y_2)

$\therefore$ $\alpha + \beta$ = (x_1+ x_2, y_1 + y_2)

$\therefore$ T ($\alpha + \beta$) = T (x_1+ x_2, y_1 + y_2)

= (3(x_1+ x_2,) + 2 (y_1 + y_2),

3(x_1+ x_2,)- 4 (y_1 + y_2))

= (3x + 2y_1,3x_1 –4y_1) + (3x_2 + 2y_2, 3x_2- 4y_2)

= T (x_1,y_1) + T (x_2,y_2)

= T (α) + T(β)

and cα = c = (x_1,, y_1)= (cx_1 ,cy_1)

$\therefore$ T (cα) = T (cx_1, cy_1)

= (3 (cx_1) + 2 (cx_1), 3(cx_1)-4 (cy_1)

= (c (3x_1 + 2y_1), c (3x_1 – 4y_1))

= c (3x_1 + 2y_1, 3x_1 – 4y_1)

= cT (x_1,, y_1)

= cT (c α)

$\therefore$ T : V_2 (R) $\rightarrow$ V_2 (R) is a L.T . and hence is a linear map on V_2 (R)

(2) Define T: R^3 $\rightarrow$ R^3 by T (x,y,z) = 2x + y ,y-z,2y + 4z). Verify whether T is a linear transformation.

***Solution* :** Let α = (x_1,y_1,z_1) , β = (x_2,y_2,z_2)

$\therefore \alpha + \beta$ = (x_1+ x_2, y_1+ y_2, z_1+ z_2)

$\therefore$ T ($\alpha + \beta$) = 2(x_1+ x_2) + (y_1+ y_2), (y_1+ y_2)

$- (z_1+z_2), 2(y_1+y_2)+4(z_1+z_2)$

$= (2x_1+y_1+2x_2+y_2, y_1-z_1+y_2-z_2, 2y_1+4z_1+2y_2+4z_2)$

$= (2x_1+y_1, y_1-z_1, 2y_1+4z_1)+(2x_2+y_2, y_2-z_2, 2y_2+4z_2)$

$= T(x_1,y_1,z_1)+T(x_2,y_2,z_2)$

$= T(\alpha)+T(\beta)$

$c\alpha = c(x_1, y_1, z_1) = (cx_1, cy_1, cz_1)$

$\therefore \quad T(c\alpha) = T(cx_1, cy_1, cz_1)$

$= (2(cx_1)+(cy_1), cy_1-cz_1, 2(cy_1)+4(cz_1))$

$= (c((2x_1+y_1, y_1-z_1, 2y_1+4z_1)$

$= cT(x_1,y_1,z_1)$

$= cT(\alpha)$

$\therefore$ T is a L.T.and hence is a linear map on R^3.

(3) T: $V_1(R) \to V_3(R)$ is defined by $T(x) = (X,2X^2,X^3)$. verify whether T is a linear transformation. (M 02)

***Solution* :** Let $\alpha = x$, $\beta = y$

$\therefore T(\alpha+\beta) = T(x+y)$

ie, $T(\alpha+\beta) = (x+y, 2(x+y)^2, (x+y)^3)$…………(1)

$T(\alpha)+T(\beta) = T(x)+T(y)$

$= (x,2x^2,x^3)+(y,2y^2,y^3)$

ie, $T(\alpha)+T(\beta) = (x+y, 2(x^2+y^2)\ x^3+y^3)$ ………….(2)

From (1) and (2) ,it is clear that

$T(\alpha+\beta) \neq T(\alpha)+T(\beta)$

Hence T is not a linear transformation.

Note : Powers of a variable can not form L.T.

(4) Define a mapping $T : V_3(F) \to V_2(F)$ by $T(a_1,a_2,a_3) = (a_2,a_3)$. Verify whether T is a linear transformation.

***Solution* :** Let $\alpha = (a_1,a_2,a_3)$, $\beta = (b_1,b_2,b_3)$

$(\alpha+\beta) = (a_1+b_1, a_2+b_2, a_3+b_3)$

$T(\alpha+\beta) = T(a_1+b_1, a_2+b_2, a_3+b_3)$

$= (a_2+b_2, a_3+b_3)$

$= (a_2,a_3)+(b_2,b_3)$

$= T(a_1,a_2,a_3)+T(b_1,b_2,b_3)$

$= T(\alpha)+T(\beta)$

$c\alpha = c(a_1,a_2,a_3) = (ca_1, ca_2, ca_3)$

$T(c\alpha) = T(ca_1, ca_2, ca_3)$

$= (ca_2, ca_3)$

$= c(a_2, a_3)$

$= cT(a_1, a_2, a_3) \qquad = cT(\alpha)$

$\therefore$ T is a linear transformation.

(5) Define the mapping T : $V_2(R) \rightarrow V_2(R)$ by T (x,y) = (xcosθ - ysinθ,xsinθ + ycosθ). Verify whether T is a linear transformation.

***Solution* :** Let $\alpha = (x_1,y_1)$, $\beta = (x_2,y_2)$

$T(\alpha + \beta) = T(x_1+x_2, y_1+y_2)$

$= ((x_1+x_2)\cos\theta - (y_1+y_2)\sin\theta, (x_1+x_2)\sin\theta + (y_1+y_2)\cos\theta)$

$= (x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta) + (x_2\cos\theta - y_2\sin\theta, x_2\sin\theta + y_2\cos\theta)$

$= T(x_1,y_1) + T(x_2,y_2)$

$= T(\alpha)+T(\beta)$

$T(c\alpha) = T(cx_1,cy_1)$

$= [(cx_1\cos\theta - cy_1\sin\theta, cx_1\sin\theta + cy_1\cos\theta)]$

$= (c(x_1\cos\theta - y_1\sin\theta), c(x_1\sin\theta + y_1\cos\theta))$

$= c(x_1\cos\theta - y_1\sin\theta, x_1\sin\theta + y_1\cos\theta)$

$= cT(\alpha)$

$\therefore$ T is a linear transformation and hence is a linear map on $V_2(R)$

(6) Let M(R) be the vector space of all 2×2 matrices over R and B be a fixed non-zero element of M(R) .

show that the mapping T: M(R) $\rightarrow$ M(R) defined by T(A) = AB+BA, $\forall$ A$\in$ M(R) , is a linear transformation .

***Solution* :** Let A,C $\in$ M(R) be any arbitrary elements.

T (A+C) = (A+C) B+B(A+C)

= AB + CB +BA + BC

= (AB+BA) + (CB+BC)

= T (A) + T(C)

Let k$\in$ R

T (k.A) = (k. A)B + B(K. A)

= k.(AB + BA)

= kT(A)

$\therefore$ T is a linear transformation.

7. Prove that if $T: V_3(R) \rightarrow V_2(R)$ is defined by

$T(x,y,z) = (x+2, y-x+2)$ **is a linear transformation**

$$T(\alpha+\beta) = T[(x_1,y_1,z_1)+(x_2,y_2,z_2)]$$

$$= T[(x_1+x_2, y_1+y_2\ z_1+z_2)]$$

$$= (x_1+x_2+z_1+z_2,\quad y_1+y_2-(x_1+x_2)+z_1+z_2)$$

$$= (x_1+z_1, y_1-x_1+z_1)+(x_2+z_2, y_2-x_2+z_2)$$

$$= T(\alpha)+T(\beta)$$

Also $T(c\alpha)=T\left[c(x,y,z)\right]=T[cx,cy,cz]$

$=(cx+cz,\,cy-cx,\,cx+cz)=c(x+z,y-x+z)$

$=c\,T(\alpha)$

$\therefore$ T is a linear transformation

8. Prove that $T:V_2(R)\to V_2(R)$ defined by $T(x,y)=(x\cos\theta+y\sin\theta,\quad x\tan\theta+y\cos\theta)$ is a linear transformation.

Solution **:** Let α (x_1, y_1) and β = (x_2 , y_2)

T (c_1 α +c_2 β) $=T\left[c_1(x_1,y_1)+c_2(x_2,y_2)\right]$

$=T[c_1x_1+c_2x_2,c_1y_1+c_2y_2]$

$=\left[c_1(x_1\cos\theta+y_1\sin\theta)+c_2(x_2\cos\theta+y_2\sin\theta),\right.$

$\left.c_1(x_1\tan\theta+y_1\cos\theta)+c_2(x_2\tan\theta+y_2\cos\theta)\right]$

$=c_1(x_1\cos\theta+y_1\sin\theta,\quad x_1\tan\theta+y_1\cos\theta)$

$c_2(x_2\cos\theta+y_2\sin\theta,\quad x_2\tan\theta+y_2\cot\theta)$

$=c_1T(x_1,y_1)+c_2T(x_2,y_2).$

$\therefore$ T is linear transformation.

Note **:** The above problem can also be done using

T$(\alpha+\beta)=T(\alpha)+T(\beta)$ *and* $T(c.\alpha)=cT(\alpha).$

9. Verify whether $T:R^3\to R^2$ defined by

$T(x,y,z)=(2x+3y,\ y+z+1)$ **is a linear transformation.**

(A 2004)

Solution : Let $\alpha=(x_1y_1z_1)\,\beta=(x_2,y_2,z_2)$

$T(\alpha+\beta)=T[(x_1+x_2,y_1+y_2,z_1+z_2]$

$=[2(x_1+x_2)+3(y_1+y_2),\quad (y_1+y_2)+(z_1+z_2)+1]$

$\neq[2x_1+3y_1,\ y_1+z_1+1]+[2x_2+3y_2,\ y_2+z_2+1]$

$\therefore$ T is not linear transformation.

Note : when a constant is present, it cannot form L.T.

10. Prove that $T=R^3\to R^2$ defined by $T(x,y,z)=(x+y,\ y+2z)$ is a linear transformation

(N 2004)

Solution **:** Let $\propto=(x_1,y_1,z_1,)\quad \beta=(x_2,y_2,z_2)$

$T[c_1\,\alpha+c_2\beta]=T\left[c_1(x_1,y_1,z_1)+c_2(x_2,y_2,z_2)\right]$

$=T[c_1x_1+c_2x_2,c_1y_1+c_2y_2,c_1z_1+c_2z_2]$

$=\left[c_1x_1+c_2x_2+c_1y_1+c_2y_2,\quad c_1y_1+c_1y_2+2(c_1z_1+c_2z_2)\right]$

$=(c_1x_1+c_1y_1,c_1y_1+2c_1z_1)+(\beta x_2+\beta y_2,\beta y_2+z\beta z_2)$

$=c_1[x_1+y_1,y_1+2z_1]+c_2[c_2x_2+c_2y_2,c_2y_2+c_2z_2]$

$=c_1T(\alpha)+c_2T(\beta)$

$\therefore$ T is a linear transformation.

1.10 properties of linear transformation.:

Theorem 1: If $T : U \to V$ is a linear transformation, then

(i) $T(0) = 0'$ where 0 and $0'$ zero vectors of U and V respectively.

(ii) $T(-\alpha) = -T(\alpha)$, $\forall \alpha \in U$

(iii) $T(C_1\alpha_1 + C_2\alpha_2 + \ldots + C_n\alpha_n)$

$$= C_1T(\alpha_1) + C_2T(\alpha_2) + \ldots + C_nT(\alpha_n)$$

***Proof* :**

(i) $\forall \alpha \in U$

$T(\alpha + 0) = T(\alpha) + T(0)$ since T is a L.T.

$\Rightarrow T(\alpha) = T(\alpha) + T(0)$

$\Rightarrow T(\alpha) + 0' = T(\alpha) + T(0)$

$\Rightarrow 0' = T(0)$ (by left cancellation law in V)

$\Rightarrow T(0) = 0'$

(ii) $T(\alpha + (-\alpha)) = T(\alpha) + T(-\alpha)$ since T is linear.

i.e $T(0) = T(\alpha) + T(-\alpha)$

i.e., $0 = T(\alpha) + T(-\alpha)$ $\because$ $T(0) = 0'$

Similarly $0' = T(-\alpha) + T(\alpha)$

$\therefore$ $T(-\alpha)$ is the additive inverse of $T(\alpha)$

i.e., $T(-\alpha) = -T(\alpha)$

(iii) Let us prove this result by Mathematical induction.

Let $P(n) : T(C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots + C_n\alpha_n)$

$$= C_1T(\alpha_1) + C_2T(\alpha_2) + \ldots\ldots + C_nT(\alpha_n).$$

If n =1 , $P(1) = T(C_1\alpha_1) = C_1T(\alpha_1)$

Since T is linear , P(1) is true.

Let n = m, $P(m) : T(C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots + C_m\alpha_m)$

$$= C_1T(\alpha_1) + C_2T(\alpha_2) + \ldots\ldots + C_mT(\alpha_m).$$

We have to show that P(m+1) is true

$T(C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots + C_{m+1}\alpha_{m+1})$

$= T(C_1\alpha_1 + C_2\alpha_2 + \ldots\ldots + C_m\alpha_m) + T(C_{m+1}\alpha_{m+1})$

$= C_1T(\alpha_1) + C_2T(\alpha_2) + \ldots\ldots + C_mT(\alpha_m)$

$+ C_{m+1}T(\alpha_{m+1})$

$\therefore$ P(m+1) is true.

Since P(1) is true and P(m) is true $\Rightarrow$ P(m+1) is true, by mathematical induction, P(n) is true for all positive integers n.

Theorem 2: If $\beta_1, \beta_2, \ldots\ldots \beta_m$ is any basis of the vector space U and $\alpha_1, \alpha_2, \ldots\ldots \alpha_m$ are any m vectors of the vector space V, then there exists one and only one linear transformation $T : U \to V$ such that $T(\beta_i) = \alpha_i$ for $i = 1, 2, \ldots\ldots m$.

Proof : Let $\alpha \in$ U be any arbitrary vector of U.

$\therefore \quad \alpha = c_1\beta_1 + c_2\beta_{2,} + \ldots\ldots\ldots + c_m\beta_m$ for

$c_1, c_2, \ldots\ldots\ldots c_m \in F$.

Define $T : U \to V$ by

$T(\alpha) = c_1\alpha_1 + c_2\alpha_2, \ldots\ldots\ldots c_m\alpha_m$.

We shall prove that this is the required L.T. for this we shall show that

(i) T is linear

(ii) $T(\beta_i) = \alpha_i$ for $i = 1,2,\ldots\ldots\ldots m$.

(iii) T is unique.

(i) Consider $\alpha, \beta \in$ U

$\therefore \quad \alpha = c_1\beta_1 + c_2\beta_{2,} + \ldots\ldots\ldots + c_m\beta_m$

$\beta = d_1\beta_1 + d_2\beta_{2,} + \ldots\ldots\ldots + d_m\beta_m$

$\alpha + \beta = (c_1 + d_1)\beta_1 + (c_2 + d_2)\beta_{2,} + \ldots\ldots + (c_m + d_m)\beta_m$

$T(\alpha + \beta) = T[(c_1 + d_1)\beta_1 + (c_2 + d_2)\beta_{2,} + \ldots + (c_m + d_m)\beta_m]$

$= (c_1 + d_1)\alpha_{1,} + (c_2 + d_2)\alpha_2, + \ldots\ldots\ldots + (c_m + d_m)\alpha_{m.}$

$= c_1\alpha_1 + d_1\alpha_1 + c_2\alpha_2 + d_2\alpha_2, + \ldots\ldots + c_m\alpha_{m.} + d_m\alpha_{m.}$

$= (c_1\alpha_1 + c_2\alpha_2, \ldots\ldots c_m\alpha_m) + (d_1\alpha_1 + d_2\alpha \ldots\ldots d_m\alpha_m)$

$= T(c_1\beta_1 + c_2\beta_{2,} + \ldots\ldots + c_m\beta_m) + (d_1\beta_1 + d_2\beta_{2,} + \ldots\ldots + d_m\beta_m)$

$= T(\alpha) + T(\beta)$

$c\alpha = c(c_1\beta_1 + c_2\beta_{2,} + \ldots\ldots\ldots + c_m\beta_m)$

$= cc_1\beta_1 + cc_2\beta_{2,} + \ldots\ldots\ldots + cc_m\beta_m$

$\therefore T(c\alpha) = T(cc_1\beta_1 + cc_2\beta_{2,} + \ldots\ldots\ldots + cc_m\beta_m)$

$= cc_1\alpha_1 + cc_2\alpha_2, \ldots\ldots\ldots cc_m\alpha_m$

$= c(c_1\alpha_1 + c_2\alpha_2, \ldots\ldots\ldots + c_m\alpha_m)$

$= cT(c_1\beta_1 + c_2\beta_{2,} + \ldots\ldots\ldots + c_m\beta_m)$

$= cT(\alpha)$

$\therefore$ T is a linear transformation.

(ii) $\beta_i = 0.\beta_1 + 0.\beta_{2,} + \ldots\ldots\ldots + 0.\beta_{i-1}$

$+ 1.\beta_1 + 0.\beta_{i+1} + \ldots\ldots + 0.\beta_m$

$\therefore T(\beta_i) = T(0.\beta_1 + 0.\beta_{2,} + \ldots\ldots\ldots + 0.\beta_{i-1}$

$+ 1.\beta_1 + 0.\beta_{i+1} + \ldots\ldots + 0.\beta_m)$

$= \quad 0\alpha_1 + 0\alpha_2 \ldots + 0\alpha_{i-1} + 1.\alpha_{i+1} + \ldots + 0\alpha_m$

$= 0.\alpha_1 + 0.\alpha_{2,} + \ldots + 0.\alpha_{i-1} + 1.\alpha\,\alpha_1 + 0._{i+1} + \ldots + 0.\alpha_m$

$= \alpha_i$

$\therefore T(\beta_i) =$ for $i = 1,2,\ldots\ldots m$.

(ii) If possible let there be another linear transformation

$S : U \to V$ such that $S(\beta_i) = \alpha_i$

$S(\alpha) = S(c_1\beta_1 + c_2\beta_2 + \ldots\ldots + c_m\beta_m)$

$= c_1S(\beta_1) + c_2S(\beta_2) + \ldots\ldots + c_mS(\beta_m)$

$= c_1\alpha_1 + c_2\alpha_2, \ldots\ldots + c_m\alpha_m$

$= T(c_1\beta_1 + c_2\beta_2 + \ldots\ldots + c_m\alpha_m)$

$= T(\alpha)$

$\therefore S(\alpha) = T(\alpha)$ or any arbitrary vector $\alpha \in U$

$\therefore S = T$. Hence L.T is unique.

Remark: From this theorem, to determine a linear transformation T from U into V, first define T on a basis of U and then extend to the remaining elements of U by expressing them as a linear combination of the basis elements. This is called linear extension of T. we shall illustrate this process in the following worked examples.

Worked examples :

(1) Find a linear transformation $T : V_2(R) \to V_2(R)$ such that T(1, 2) = (3, 0) and T (2, 1) = (1, 2)

Solution : Let us express (1, 2) and (2,1) as linear combination of the basis vector $e_1 = (1,0)$ and $e_2 = (0,1)$.

$(1,2) = 1(1,0) + 2(0,1) = 1e_1 + 2e_2$

$(2,1) = 2(1,0) + 1(0,1) = 2e_1 + 1e_2$.

$\therefore T(e_1 + 2e_2) = T(1,2)$ and $T(2e_1 + 1e_2) = T(2,1)$

ie, $T(e_1) + 2T(e_2) = (3,0)$ …………..(1)

and $2T(e_1) + T(e_2) = (1.2)$ …………….(2)

Solve (1) and (2) for $T(e_1)$ and $T(e_2)$

Multiply (2) by 2 and subtract from (1)

We get $-3\,T(e_1) = (3, 0) - (2, 4)$

ie,- $3T(e_1) = (1, -4)$

$\therefore \quad T(e_1) = \left(\frac{-1}{3}, \frac{4}{3}\right)$

From (2) $T(e_2) = (1,2) - 2\,T(e_1)$

$= (1,2) - 2\left(\frac{-1}{3}, \frac{4}{3}\right)$

$= \left(\frac{5}{3}, \frac{-2}{3}\right)$

Now $T(x,y) = T[x(1,0) + y(0,1)]$

$= T[x\,e_1 + y\,e_2]$

$= xT(e_1) + y\,T(e_2)$

$= x\left(\frac{-1}{3}, \frac{4}{3}\right) + y\left(\frac{5}{3}, \frac{-2}{3}\right)$

$= \left(\frac{-x}{3} + \frac{5y}{3}, \frac{4x}{3} - \frac{2y}{3}\right)$

ie, T(x,y) = $\left(\frac{-x+5y}{3}, \frac{4x-2y}{3}\right)$ is the required linear transformation.

(2) Find a linear transformation T: V_3(R) $\rightarrow$ V_2(R) such that T (1,0,0) = (-1,0), T (0,1,0) = (1,1),T (0,0,1) = (0,-1)

***Solution* :** e_1= (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)

$\therefore$ T (e_1) = (-1,0), T(e_2) (1,1),T(e_3)= (0,-1)

Now, (x, y, z) = x(1,0,0)+ y(0,1,0) + z(0,0,1)

$\therefore$ T (x,y,z) = T[x(1,0,0)+y(0,1,0)+z(0,0,1)]

= xT(e_1) + yT (e_2)+zT(e_3)

= x (-1,0)+ y (1,1) + z (0,-1)

= (-x + y ,y-z)

ie, T(x,y,z) = (y-x,y-z)

(3) Find a linear transformation T:$R^2 \rightarrow R^3$ such that

T (-1,1) = (-1,0,2) and T (2,1) = (1,2,1)

***Solution* :** Let us express (-1,1)and (2,1) as linear combination of e_1= (1,0) and e_2 = (0,1)

(-1,1) = - 1 (1,0) + 1(0,1) = - e_1+ e_2

(2,1) = 2 (1,0) + 1(0,1) = 2 e_1+ e_2

$\therefore$ T (-e_1+ e_2) = T (-1,1) and T(2 e_1+ e_2) = T (2,1).

i.e, - T (e_1) +T (e_2) + = (-1, 0, 2) ....(1)

2T (e_1) + T(e_2) = (1, 2, 1) ....(2)

Solve equation (1) and (2) for T(e_1) and T(e_2).

Subtracting (1) from (2) weget

3T(e_1) = (2, 2, -1) $\Rightarrow$ $\therefore$ T(e_1) = $\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

Subtracting in (1) we get

T(e_2) = (-1, 0, 2)+ $\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$

i.e., T(e_2) = $\left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$

T (x,y) = T [x(e_1) + y(e_2)]

= xT (e_1) + yT (e_2)

= x $\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$+y$\left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$

= $\left(\frac{2x-y}{3}, \frac{2x+2y}{3}, \frac{-x+5y}{3}\right)$

$\therefore$ T(x,y) = $\left(\frac{2x-y}{3}, \frac{2(x+y)}{3}, \frac{5y-x}{3}\right)$

(4) Find a linear transformation T : $R^3 \rightarrow R^3$ such that

T (1,1,1) = (1,1,1), T (1,2,3)= (-1,-2,-3) and T (1,1,2) = (2,2,4).

Solution **:** Let us express (1,1,1), (1, 2, 3),(1,1, 2) as linear combinations of $e_1 = (1, 0, 0)$,$e_2 = (0,1, 0)$,$e_3 = (0, 0, 1)$

$$(1,1,1) = 1\,e_1 + 1\,e_2 + 1\,e_3$$

$$(1,2,3) = 1\,e_1 + 2\,e_2 + 3\,e_3$$

$$(1,1,2) = 1\,e_1 + 1\,e_2 + 2\,e_3$$

$$T(1,1,1) = T(e_1 + e_2 + e_3)$$

$$T(1,2,3) = T(e_1 + e_2 + e_3)$$

$$T(1,1,2) = T(e_1 + e_2 + e_3)$$

ie, $T(e_1) + T(e_2) + T(e_3) = (1,1,1)$ …(1)

$T(e_1) + 2T(e_2) + 3T(e_3) = (-1,-2,-3)$ ...(2)

$T(e_1) + T(e_2) + 2T(e_3) = (2,2,4)$ . . .(3)

Solve the equation (1), (2) and (3) for $T(e_1)$, $T(e_2)$ and $T(e_3)$

We get $T(e_1)$, = (4,5,8), $T(e_2)$ = (-4-5,-10), $T(e_3)$= (1,13)

Now $(x,y,z) = x\,e_1 + y\,e_2 + z\,e_3.$

$\therefore\ T(x,y,z) = x\,T(e_1) + T(e_2) + T(e_3)$

$= x(4, 5, 8) + y(-4, -5, -10) + z(1, 1, 3)$

ie, $T(x, y, z) = (4x - 4y + z, 5x - 5y + z, 8x - 10y + 3z)$

is required linear transformation.

(5) If V is the vector space of all polynomial over R, show that the mapping f: V $\to$ V defined by f(p) = p(0) isa linear map.

Solution **:** Let $p, q \in V$ $\quad \therefore\ p + q \in V$

$f(p + q) = (p + q)(0) = p(0) + q(0)$

$= f(p) + f(q)$

$f(cp) = (cp)(0) = c(p(0)) = c\,f(p)$

$\therefore\ f : V \to V$ is a linear map.

(6) If $T:R^2 \to R^2$ is a linear transformation such that T (1,0) = (1,1) and T(0,1) = (-1,2) . Show that T maps the square with vertices (0,0) , (1,0) , (1,1) , (0,1) into a parallelogram.

Solution : T (1,0) = (1,1) ,T (0,1)= (-1,2)

Let $(x,y) \in R^2$.

$\therefore\ (x,y) = x(1,0) + y(0,1)$

$\therefore\ T(x.y) = x\,T(1,0) + y\,T(0,1)$

$= x(1,1) + y(-1,2)$

$= (x - y, x + 2y)$

$\therefore\ T(x.y) = (x - y, x + 2y)$

$T(0,0) = (0,0) \equiv A$

$T(1,0) = (1,1) \equiv B$

$T(1,1) = (0,3) \equiv C$

$T(0,1) = (-1,2) \equiv D$

$\therefore$ A,B,C,D are the vertices of a quadrilateral.

To show that ABCD is a parallelogram, we have to show that the diagonals AC and BD bisect each other.

$$\text{Mid point of AC} = \left(\frac{0+0}{2}, \frac{0+3}{2}\right) = \left(0, \frac{3}{2}\right)$$

$$\text{Mid point of BD} = \left(\frac{1-1}{2}, \frac{1+2}{2}\right) = \left(0, \frac{3}{2}\right)$$

$\therefore$ Diagonals bisect each other

$\therefore$ ABCD is a parallelogram.

1.11 Matrix of a linear transformation

In this section we shall study how to associate a matrix to a linear transformation and conversely how to associate a linear transformation to a matrix.

Let U and V be two vector spaces of dimensions m and n respectively.

Let $B_1 = \{\alpha_1, \alpha_2, \ldots\ldots \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, \ldots\ldots \beta_n\}$

be the bases of U and V respectively.

Let $T: U \rightarrow V$ be a linear transformation defined

By $T(\alpha_i) = C_{i1}\beta_1 + C_{i2}\beta_2 + \ldots\ldots\ldots + C_{in}\beta_n$.

$T(\alpha_1) = C_{11}\beta_1 + C_{12}\beta_2 + \ldots\ldots\ldots + C_{1n}\beta_n$.

$T(\alpha) = C_{21}\beta_1 + C_{22}\beta_2 + \ldots\ldots\ldots + C_{2n}\beta_n$.

………………………………………………………

………………………………………………………

$T(\alpha_m) = C_{m1}\beta_1 + C_{m2}\beta_2 + \ldots\ldots\ldots + C_{mn}\beta_n$.

The coordinates of $T(\alpha_i)$, $i = 1,2, \ldots..m$ w.r.t the basis β_2 of V determine an $m \times n$ matrix

$$A = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix}$$

This matrix A is called the matrix of linear transformation T relative to the bases B_1 and B_2 . conversely , given a matrix $A = (C_{ij})_{mxn}$, we shall associate a linear transformation $T: U \rightarrow V$ where U and V are vector spaces of dimensions m and n respectively.

Consider the bases $B_1 = \{\alpha_1, \alpha_2, \ldots\ldots \alpha_n\}$and

$B_2 = \{\beta_1, \beta_2 \ldots.. \beta_n\}$ of U and V respectively.

We shall define a linear transformation $T: U \rightarrow V$ by defining the values of T on the vectors of B_1 as:

$T(\alpha_1) = C_{11}\beta_1 + C_{12}\beta_2 + \ldots\ldots\ldots + C_{1n}\beta_n$.

$T(\alpha) = C_{21}\beta_1 + C_{22}\beta_2 + \ldots\ldots\ldots + C_{2n}\beta_n$.

………………………………………………………

………………………………………………………

T(α m) = $C_{m1}\beta_1 + C_{m2}\beta_2 + \ldots\ldots\ldots\ldots + C_{mn}\beta_n$.

Now we extend T linearly to the entire space V. Further the linear transformation T is unique. Hence every matrix can be associated to a linear transformation.

We shall illustrate these in the following examples.

Worked examples:

1. Find the coordinates of the vector $\alpha = (4,2)$ belongs to R^3 relative to the ordered pair $B = \{(1,1)(3,1)\}$

Let $(4,2) = a_1(1,1) + b_1(3,1)$

$\Rightarrow 4 = a_1 + 3b_1 \quad and \quad 2 = a_1 + b_1$

$\Rightarrow a_1 = 1, \quad b_1 = 1$

$\therefore$ (1,1) is the relative bases.

2. Find the matrix of the linear transformation $T = R^2 \to R^3$ defined by $T(x,y) = (x+y,\ y+2\ z+x)$

Solution : Let $e_1 = (1,0), \quad e_2 = (0,1) \quad \in R^2$

$$f_1 = (1,0,0), \quad f_2 = (0,1,0), \quad f_3 = (0,0,1) \in R^3$$

$$T(e_1) = T(1,0) = (1,0,1) = 1f_1 + 0f_2 + 1f_3$$

$$T(e_2) = T(0,1) = (1,1,0) = 1f_1 + 0f_2 + 1f_3$$

The matrix linear transformation of

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad is \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

3. If $T : V_3(R) \to V_2(R)$ is defined by $T(x,y,z) = (y-x, y-z)$

Find matrix of T **(N 2001)**

Solution : Let $e_1, \quad e_2, \quad e_3, \quad \in \quad V_3(R)$

$$T(e_1) = T(1,0,0) = (0-1, 0-0) = (-1,0)$$
$$T(e_2) = T(0,1,0) = (1-0, 1-0) = (1,1)$$
$$T(e_3) = T(0,0,1) = (0-0, 0-1) = (0,-1)$$

$\therefore$ The matrix of the L.T is

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \quad is \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

4. Find the matrix of the linear transformation $T : R^3 \to R^3$ defined by $T(x,y,z) = (x-y+z, 2x-z, x+y-2z)$

***Solution* :** Let $e_1, e_2, e_3 \in R^3$ (M 1999)

$$T(e_1) = T(1,0,0) = (1,2,1)$$
$$T(e_2) = T(0,1,0) = (-1,0,1)$$
$$T(e_3) = T(0,0,1) = (1,-1,-2)$$

$\therefore$ The matrix of the L.T is

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix} \; is \; \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & -2 \end{bmatrix}$$

5. Find the matrix of the L.T ., $T: R^3 \to R^2$ de defined by

$T(x,y,z) = (2x+3y,\ y+2z)$ **w.r.t standard bases**

Solution : Let $e_1, e_2, e_3 \in R^3$, the standard bases (A 2004)

$T(e_1) = (1,0,0) = (2,0)$

$T(e_2) = (0,1,0) = (3,1)$

$T(e_3) = (0,0,1) = (0,2)$

The matrix linear transformation is

$$\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} \; is \; \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

6. Find the matrix of the linear transformation $T: R^3 \to R^2$

Defined by $T(x,y) = (2x+y,\ x-2y)$ **(N 04)**

Solution : Let e_1, $e_2 \in R^2$

$T(e_1) = T(1,0) = (2,1)$; $T(e_2) = T(0,1) = (1,-2)$

The matrix L.T. is $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \; is \; \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

7. Find the matrix of the linear transformation T : V_2(R) $\to$ V_2(R) defined by T (x,y) = (x,-y) w.r.t the standard basis of V_2(R).

***Solution* :**

T (x,y) = (x,-y)

T (e_1) = T (1,0) = (1,0)

T (e_2) = T (0,1) = (0,-1)

The matrix of linear transformation is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

8. Find the matrix of the linear transformation T: V_2(R) $\to$ V_3(R)such that T (-1, 1) = (-1, 0, 2) and T (2, 1) = (1, 2, 1). (M 2000)

***Solution* :** (-1,1) = - 1e_1 + 1e_2

(2,1) = 2e_1 + 1e_2

T (-1,1) = T (-e_1 + e_2) and T (2,1) = T (2e_1+ e_2)

i.e., -T (e_1) +T(e_2) = (-1,0,2) ………..(1)

2T (e_1) + T(e_2) = (1,2,1) …………(2)

Solve these equation for T(e_1) and T(e_2).

Subtracting (1) and (2) we get

3T (e_1) = (2,2,-1)

$\therefore \quad T(e_1) = \left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right)$

$\therefore T(e_2) = (-1,0,2) + \left(\frac{2}{3},\frac{2}{3},\frac{-1}{3}\right) = \left(\frac{-1}{3},\frac{2}{3},\frac{5}{3}\right)$

The matrix of L.T .is

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{5}{3} \end{bmatrix}$$

9. Find the matrix of the linear transformation

T:V_3(R) $\rightarrow V_2$(R) defined by T(x,y,z) = (x + y,y + z) w.r.t standard basis.

***Solution* :** the standard basis are (1,0,0),(0,1,0),(0,0,1).

$\therefore$ T (1,0,0) = (1 + 0,0 + 0) = (1, 0)

T (0, 1, 0) = (0 + 1, 1 + 0) = (1, 1)

T (0, 0, 1) = (0 + 0, 0 + 1) = (0, 1)

$\therefore$ The matrix of the linear transformation is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

10. Find the matrix of the linear transformation T:$R_4 \rightarrow R_3$ defined by T(x_1, x_2, x_3, x_4) = (x_1+ x_2 + $2x_3$ + $3x_4$,x_1 + x_3 –x_4, x_1+ $2x_2$) w.r.t the bases B_1 = (1, 1, 1, 2), (1,-1, 0, 0), (0, 0, 1, 1), ((0,1,0,0) and B_2 = { (1, 2, 3), (1, -1, 1),(2, 1, 1)}

***Solution* :** B_1 = {(1,1,1,2) ,(1,-1,0,0). (0,0,1,1),((0,1,0,0)}

B_2 = { (1,2,3), (1,-1,1),(2,1,1)}

T (1,1,12) = (1+1+2+6, 1 + 1-2,1 + 2) = (10, 0, 3)

T (1,-1,0,0) = (1-1 +0 +0,1+0-0,1-2) = (0,1,-1)

T (0,0,1,1) = (0+ 0 +2 + 3,0 +1-1,0 + 0) = (5,0,0)

T (0,1,0,0) = (0 +1 + 0 +0,0+0-0,0+2) = (1,0,2)

Now (10, 0, 3) = a(1, 2, 3) + b (1,-11) + c (2,1,1)

= (a + b +2c, 2a-b + c, 3a + b+c)

$\therefore$ a+b + 2c = 10, 2a-b + c = 0.3a + b +c = 3

Solving for a, b, c, we get a = $\frac{-11}{9}, b = \frac{19}{9}, c = \frac{41}{9}$.

$\therefore (10, 0, 3) = \frac{-11}{9}(1,2,3) + \frac{19}{9}(1,-1,1) + \frac{41}{9}(2,1,1)$

(0, 1, -1) = a (1,2,3) + b (1,-1,1) + c(2,1,1)

= (a +b + 2c,2a-b+c,3a + b +c)

$\therefore$ a +b +2c = 0, 2a –b + c = 1, 3a + b +c = 2

Solving these equation ,we get

a $= \dfrac{-2}{9}, b = \dfrac{-8}{9}, c = \dfrac{5}{9}.$

$\therefore$ (0,1,-1) $= \dfrac{-2}{9}(1,2,3) - \dfrac{-8}{9}(1,-1,1) + \dfrac{5}{9}(2,1,1)$

Similarly we have

(5,0,0) = (a +b+ 2c,2a - b + c,3a + b + c)

$\therefore$ a + b +2c= 5, 2a-b+c = 0, 3a + b + c = 2

Solving these equations we get a $= \dfrac{-10}{9}, b = \dfrac{5}{9}, c = \dfrac{25}{9}$

$\therefore$ (5,0,0) $= \dfrac{-10}{9}(1,2,3) + \dfrac{5}{9}(1,-1,1) + \dfrac{25}{9}(2,1,1)$

and (1, 0, 2) = (a + b + 2c, 2a - b + c, 3a + b + c)

$\therefore$ a + b +2c = 1, 2a - b + c = 0, 3a + b+ c = 2

Solving these equations we get a $= \dfrac{4}{9}, b = \dfrac{7}{9}, c = \dfrac{-1}{9}.$

$\therefore$ (1,0,2) $= \dfrac{4}{9}(1,2,3) + \dfrac{7}{9}(1,-1,1) - \dfrac{1}{9}(2,1,1)$

$\therefore$ The matrix of the linear transformation is

$$\frac{1}{9}\begin{bmatrix} -11 & 19 & 41 \\ -2 & -8 & 5 \\ -10 & 5 & 25 \\ 4 & 7 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -11 & -2 & -10 & 4 \\ 19 & -8 & 5 & 7 \\ 41 & 5 & 25 & -1 \end{bmatrix}$$

11. Find the linear transformation for the matrix A= $\begin{bmatrix} -1 & 0 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$ **with respect to**

(i) B_1 = {(1,0,0),(0,1,0),(0,0,1) and B_2 = {(1,0),(0,1)} and

(ii) B_1 = {(1,2,0),(0,-1,0)(1,-1,1) and B_2 = {(1,0),(2,-1)}

***Solution* :** (i) the given bases are:

B_1 = {(1,0,0),(0,1,0),(0,0,1) B_2 = {(1,0),(0,1)}

The matrix is A= $\begin{bmatrix} -1 & 0 \\ 2 & 0 \\ 1 & 3 \end{bmatrix}$

Define the linear transformation.

T:V_3(R) $\rightarrow$ V_2(R) by

T (1,0,0) = (-1) (1,0) + 0 (0,1) = (-1,0)

T(0,1,0) = 2(1,0) + 0(0,1) = (2,0)

T (0,0,1) = 1 (1,0)+ 3 (0,1) = (1,3)

$\therefore$ T (e_1) = (-1,0)

T (e_2) = (2,0)

T (e_3) = (1,3)

Now T (x,y,z) = T (x e_1+ y e_2+ z e_3)

$= x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$= x(-1,0) + y(2,0) + z(1,3)$

$= (-x + 2y + z, 0+0 + 3z)$

$= (-x + 2y + z, 3z)$

ie, $T(x,y,z) = (-x + 2y + z, 3z)$

(ii) The bases are $B_1 = \{(1,2,0),(0,-1,0),(1,-1,1)\}$

$B_2 = \{(1,0),(2,-1)\}$.

Define the L.T : $V_3(R) \rightarrow V_2(R)$ by

$T(1,2,0) = (-1)9(1,0) + 0(2,-1) = (-1,0)$

$T(0,-1,0) = 2(1,0) + 0(2,-1) = (2,0)$

$T(1,-1,1) = 1(1,0) + 3(2,1) = (7,-3)$

Now $(1,2,0) = 1(1,0,0) + 2(0,1,0) + 0(0,01)$

$\therefore T(1,2,0) = T(1e_1 + 2e_2 + 0e_3)$

ie, $T(e_1) + 2T(e_2) = (-1,0)$ ………….(1)

$(0,-1,0) = 0(1,0,0) - 1(0,1,0) + 0(0,0,1)$

$= 0e_1 - 1e_2 + 0e_3$

$\therefore T(0,-1,0) = T(0e_1 - 1e_2 + 0e_3)$

ie, $-T(e_2) = (2,0)$ …………..(2)

$(1,-1,1) = 1(1,0,0) - 1(0,1,0) + 1(0,0,1)$

$= 1e_1 - 1e_2 + 1e_3$

$\therefore T(1,-1,1) = T(e_1 - e_2 + e_3)$

ie, $T(e_1) - T(e_2) + T(e_3) = (7, -3)$ ……….(3)

Solving the equations (1) ,(2) and (3) ,we get from (2)

$T(e_2) = (-2, 0)$

from (1) $T(e_1) + 2T(e_2) = (-1,0)$

$\therefore \quad T(e_1) = (-1,0) - (-4,0) = (3, 0)$

from (3), we get $(3, 0) + (2, 0) + T(e_1) = (7, -3)$

$\therefore \quad T(e_1) = (7, -3) - (3, 0) - (2, 0)$

ie, $T(e_3) = (2, -3)$

$\therefore \quad T(x,y,z) = T(xe_1 + ye_2 + z\,e_3)$

$= xT(e_1) + yT(e_2) + zT(e_3)$

$= x(3,0) + y(-2,0) + z(2,-3)$

$= (3x - 2y + 2z, -3z)$

(12) Find the linear transformation or the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \textbf{w.r.t}$$

(i) **Standard bases $B_1 = B_2 = \{e_1, e_2, e_3\}$**

(ii) **$B_1 = B_2 = \{(0,1,-1),(-1,1,0),(-1,-1,0)\}$**

Solution : (i) Bases $B_1 = B_2 = \{e_1, e_2, e_3\}$

Define the L.T ; $T: V_3(R) \to V_3(R)$ by

$T(e_1) = T(1,0,0) = 0e_1 - 1e_2 + 1e_3 = (0,1,-1)$

$T(e_2) = T(0,1,0) = 1e_1 + 0e_2 + e_3 = (1,0,0)$

$T(e_3) = T(0,0,1) = 1e_1 - 1e_2 + 0e_3 = (1,-1,0)$

$\therefore T(e_1) = (0,1,-1), T(e_2) = (1,0,0), T(e_3) = (1,-1,0)$

$T(x,y,z) = T(xe_1 + ye_2 + ze_3)$

$= x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$= x(0,1,-1) + y(1,0,0) + z(1,-1,0)$

ie, $T(x,y,z) = (y + z, x-z, -x)$ is the required linear transformation.

(ii) Bases $B_1 = B_2 = \{(0,1,-1),(-1,1,0),(-1,-1,0)\}$

Define the L.T $T:V_3(R) \to V_3(R)$ by

$T(0,1,-1) = 0(0,1,-1) + 1(-1,1,0) - 1(-1,-1,0)$

$= (0,2,0)$

$T(-1,1,0) = 1(0,1,-1) + 0(-1,1,0) + 0(-1,-1,0)$

$= (0,1,-1)$

$T(-1-10) = 1(0,1,-1) - 1(-1,1,0) + 0(-1,-1,0)$

$= (1,0,-1)$

Now $(0,1,-1) = 0e_1 - 1.e_2 - 1e_3$

$\therefore T(0,1,-1) = 0\,T(e_1) + 1\,T(e_2) - 1\,T(e_3)$

ie, $T(e_2) - T(e_3) = (0,2,0)$ …………(1)

$(-1,1,0) = -1.e_1 + 1.e_2 + 0.e_3$

$\therefore T(-1,1,0) = -T(e_1) + T(e_2) + 0\,T(e_3)$

ie, $-T(e_1) + T(e_2) = (0,1,-1)$ ………..(2)

$(-1,-1,0) = -1.e_1 - 1.e_2 + 0.e_3$

$\therefore T(-1,-1,0) = -T(e_1) - T(e_2) + 0\,T(e_3)$

ie, $-T(e_1) - T(e_2) = (1,0,-1)$ ………..(3)

Solving equation (1), (2), (3) we get

$T(e_1) = \left(\frac{-1}{2}, \frac{-1}{2}, 1\right)$, $T(e_2) = \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$,

$T(e_3) = \left(\frac{-1}{2}, \frac{-3}{2}, 0\right)$

$\therefore T(x,y,z) = T(xe_1 + ye_2 + ze_3)$

$= x\left(\frac{-1}{2}, \frac{-1}{2}, 1\right) + y\left(\frac{-1}{2}, \frac{1}{2}, 0\right) + z\left(\frac{-1}{2}, \frac{-3}{2}, 0\right)$

i.e $T(x,y,z) = \left(-\frac{x}{2} - \frac{y}{2} - \frac{z}{2}, -\frac{x}{2} + \frac{y}{2} - \frac{3z}{2}, x\right)$ is the required

13. Find the linear transformation $T: R^3 \to R^2$ corresponding to the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}$ w.r.t the bases

$B_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$ **to** $B_2 = \{(2,1),(3,1)\}$ **(A 04)**

***Solution*:** $T(1,0,0) = 1(2,1) - 1(3,1) = (-1,0)$

$$T(0,1,0) = 2(2,1) + 1(3,1) = (7,3)$$

$$T(0,0,1) = 3(2,1) - 0(3,1) = (6,3)$$

Let $(x,y,z) \in R^3$

$$(x,y,z) = C_1(1,0,0) + C_2(0,1,0) + C_3(0,0,1) = (C_1,C_2,C_3)$$

$$\therefore \quad C_1 = x, \quad C_2 = y, \quad C_3 = z$$

$$\therefore \quad (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

$$T(x,y,z) = xT(1,0,0)\, yT(0,1,0)\, zT(0,0,1)$$

$$= x(-1,0)\, y(7,3)\, z(6,3)$$

$$= (-x + 7y + 6z\,,\, 3y + 3z)$$

14. If the matrix of linear transformation T on $V_2(R)$ relative to standard basis of $V_2(R)$ is $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$ then what is the matrix of T relative to the basis $B_1 = \{(1,1),(1,-1)\}$

***Solution* :** Let e_1 & e_2 are standard basis of $V_2(R)$ for the given

Matrix $T(1,0) = 2(1,0) + 1(0,1) = (2,1)$

$$T(0,1) = -3(1,0) + 1(0,1) = (-3,1)$$

For $(x, y) \in V_2(R)$

$$T(x,y) = [x(1,0) + y(0,1)] = x(2,1) + y(-3,1)$$

$$= (2x - 3y, \quad x + y)$$

From the given basis B_1,

$$T(1,1) = (2-3, 1+1) = (-1,2) = \frac{1}{2}(1,1) - \frac{3}{2}(1,-1)$$

$$T(1,-1) = (2+3, 1-1) = (5,0) = \frac{5}{2}(1,1) + \frac{5}{2}(1,-1)$$

$$\therefore \quad \text{Required matrix Transformation is } = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} \\ \frac{-3}{2} & \frac{5}{2} \end{bmatrix}$$

15. For the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ find the corresponding linear transformation $T : R^2 \to R^2$ w.r.t the bases $\{(1,0),(1,1)\}$

Let $(x,y) \in R^2$, $\quad (x,y) = a_1(1,0) + b_1(1,1)$

$$x = a_1 + b_1, \quad y = b, \quad \therefore a_1 = x - y,$$

$$\Rightarrow (x,y) = (a+b, b)$$

$$= (x-y, y)$$

$$\Rightarrow (x,y) = (x-y)(1,0) + y(1,1)$$

$$T(x,y) = (x-y)T(1,0) + yT(1,1)$$

$$= (x-y)(4,3) + y(6,4)$$

$$=(4x-4y+6y, \quad 3x-3y+4y)$$

$$=(4x+2y, \quad 3x+y)$$

This is the matrix transformation.

16. Find the matrix of the linear transformation

$T:V_3(R)\to V_2(R)$ **defined by** $T=(x+y, \quad y+z)$ **relative**

to bases (i) standard bases of $V_3(R)$ *and* $V_2(R)$

(ii) $B_1=\{(1,1,1),(1,0,0)(1,1,0)\}$ *of* V_3R

(iii) $B_2=\{(1,0) \quad (0,1)\}$ *of* $V_2(R)$

Solution **:** case (i) e_1, e_2, e_3 are the standard bases of V_3 (R) & e_1 & e_2 are of V_2 (R)

$$T(e_1)=T(1,0,0)=(1,0)=1(1,0)+0(0,1)$$

$$T(e_2)=T(0,1,0)=(1,1)=1(1,0)+1(0,1)$$

$$T(e_3)=T(0,0,1)=(0,1)=0(1,0)+1(0,1)$$

Thus the matrix linear transformation is

$$\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix} \quad is \quad \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}$$

(ii) $T(1,1,1)=(2,2)=2(1,0)+2(0,1)$

$$T(1,0,0)=(1,0)=1(1,0)+0(0,1)$$

$$T(1,1,0)=(1,1)=1(1,0)+1(0,1)$$

Thus the matrix linear transformation is

$$\begin{bmatrix}2 & 2\\ 1 & 0\\ 1 & 1\end{bmatrix} \quad is \quad \begin{bmatrix}2 & 1 & 1\\ 2 & 0 & 1\end{bmatrix}$$

17. Find the matrix of the linear transformation $T:R^2\to R^3$ **defined**

by $T(x,y)=(2y-x, \quad y, \quad 3y-3x)$ **relative to bases**

$B_1=\{(1,1)(-1,1)\}$ *and* $B_2=\{(1,1,1),(1,-1,1)(0,0,1)\}$

Solution : $T(1,1)=(1,1,0)=a_1(1,1,1),+a_2(1,-1,1)+a_3(0,0,1)$

$$\Rightarrow a_1+a_2=1 \quad a_2=0, \quad a_1+a_2+a_3=0$$

$$\Rightarrow a_1=1 \quad a_2=0, \quad a_3=-1$$

$$T(-1,1)=(3,1,6)=b_1(1,1,1)+b_2(1,-1,1)+b_3(0,0,1)$$

$$\Rightarrow b_1+b_2=3, \quad b_1-b_2=1, \quad b_1-b_2+b_3=6$$

$$\Rightarrow b_1=2, \quad b_2=1, \quad b_3=3$$

$\therefore$ Matrix L.T. is $\begin{bmatrix}1 & 0 & -1\\ 2 & 1 & 3\end{bmatrix}$ *is* $\begin{bmatrix}1 & 2\\ 0 & 1\\ -1 & 3\end{bmatrix}$

EXERCISE.

1. Find the coordinates of the vector $\alpha\in R^n$ **relative to the ordered basis mentioned**

a) $\alpha=(3,-4) \quad B=\{(1,0),(0,1)\}$

b) $\alpha = (-4,-1,2) \quad B = \{(1,1,1)(1,2,3)(1,0,0)\}$

2. Find the matrix of the following transformation :

(i) $T: V_2(R) \to V_2(R)$ defined by $T(x,y) = (3x, \ x-y)$

(ii) $T: V_3(R) \to V_3(R)$ defined by $T(x,y,z) = (z-2y, \ x+2y-z)$

(iii) $T: V_3(R) \to V_3(R)$ defined by $T(x,y,z) = (x+y, 2y \ 2y-x)$

(iv) $T: R^2 \to R^3$ defined by $T(x,y) = (3x-y, \ 2x+4y, \ 5x-6y)$

(v) $T: R^2 \to R^3$ defined by $T(x,y,z) = (x+2y-z,$

$y+z, \ x+y-2z)$

(vi) $T: R^3 \to R^2$ defined by $T(x,y,z) = (3x-2y+z, x-3y-2z)$

3. Find the matrix for the following

a) $T: R^2 \to R^2$ defined by $T(-1,1) = (-1,0,2) \ \ T(2,1) = (1,2,1)$

b) $T: R^2 \to R^2$ defined by $T(2,1) = (3,4) \ \ T(-3,4) = (0,5)$

4. Find the matrix of the following :-

(i) $T: R^3 \to R^2$ defined by $T(e_1) = 2f_1 - 2f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0$ where $\{e_1, e_2 e_3\}$ and $\{f_1, f_2\}$ are standard basis of R^3 and R^2

(ii) $T: R^3 \to R^3$ defined by $T(1,1,1) = (1,1,1) T(1,2,3) = (-1,2,3)$ and $T(1,1,2) = (2,2,4)$

(iii) $T: R^2 \to R^3$ defined by $T(-1,1) = (-1,0,2), T(2,1) = (1,2,1)$

5. Find the matrix of the following transformation

(i) $T: R^3 \to R^2$ defined by $T(x,y,z) = (2y+z, x-4y, 3x)$ w.r.t the bases $\{(1,1,1),(1,1,0),(1,0,0)\}$

(ii) $T: R^2 \to R^3$ defined by $T(x,y) = (-x+2y, y, 3y-3x)$ w.r.t. the bases $B_1 = \{(1,2)(-2,1)\}, B_2 = \{(-1,0,2),(1,2,3),(1,-1,-1)\}$

(iii) $T: R^2 \to R^3$ defined by $T(x,y) = (x,y,0)$ w.r.t the standard bases

(iv) $T: R^2 \to R^2$ defined by $T(x,y) = (x+4y, 2x-3y)$ w.r.t $B_1 = \{e_1, e_2\} \ B_2 = \{(1,3)(2,5)\}$

6. For the matrix and the bases find the matrix transformation

(i) $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & 4 & 1 & -4 \end{bmatrix}'$ w.r.t the standard bases

(ii) $\begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 0 \end{bmatrix}'$ **w.r.t the standard bases of R^3 and R^2**

7. For the following matrices and bases, determine the linear transformation, such that the matrix is the matrix of T w.r.t the bases

(i) $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & 3 \end{bmatrix}$ w.r.t $\quad B_1 = \{(-2,1)(1,2)\}$

$B_2 = \{(1,-1,-1),(1,2,3),(-1,0,2)\}$

(ii) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ w.r.t a) Standard bases

b) $B_1=\{(1,1,1)(1,0,0)(0,1,0)\}$

$B_2=\{(1,2,3)(1,-1,1)(2,1,1)\}$

(iii) $\begin{bmatrix}1&0&-1\\2&1&3\end{bmatrix}$ w.r.t $B_1=\{(1,1)(-1,1)\}$

$B_2=\{(1,1,1)(1,-1,1)(0,0,1)\}$

Answers

1) $a)\ \left(-\frac{1}{2}\ \ \frac{7}{2}\right)$ $\quad b)\ (-7,3,0)$

2) (i) $\begin{bmatrix}3&1\\0&-1\end{bmatrix}$ (ii) $\begin{bmatrix}0&1\\-2&2\\1&1\end{bmatrix}$ (iii) $\begin{bmatrix}1&0&-1\\1&2&2\\0&0&0\end{bmatrix}$

(iv) $\begin{bmatrix}3&2&5\\-1&4&6\end{bmatrix}$ (v) $\begin{bmatrix}1&0&1\\2&1&1\\-1&1&2\end{bmatrix}$ (vi) $\begin{bmatrix}3&1\\-2&-3\\1&-2\end{bmatrix}$

3) a) $\frac{1}{3}\begin{bmatrix}2&2&-1\\-1&2&5\end{bmatrix}$ b) $\frac{1}{11}\begin{bmatrix}12&11\\9&22\end{bmatrix}$

4) (i) $\begin{bmatrix}2&-1\\1&2\\0&0\end{bmatrix}$ (ii) $\begin{bmatrix}4&1&8\\-4&-1&-10\\1&1&3\end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix}2&2&-1\\-1&2&5\end{bmatrix}$

5) (i) $\begin{bmatrix}3&-6&6\\3&-6&5\\2&-2&-1\end{bmatrix}$ (ii) $\begin{bmatrix}-1&0&1\\3&1&2\end{bmatrix}$ (iii) $\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$

(iv) $\begin{bmatrix}-1&-2&6\\1&1&5\end{bmatrix}$

6) (i) $T(x,y,z,t)=(x-y+2t,\quad 3x+4y+2z-4t)$

(ii) $T(x,y,z)=(x-y-2z,\quad 3x+y)$

7) (i) $T(x,y)=\left(\frac{2x+4y}{5},\frac{-x-2y}{5},\frac{-17x+y}{5}\right)$

$T(x,y)=\left(\frac{7x+9y}{5},\quad x\quad ,\frac{-12x-4y}{5}\right)$

(ii) a) $T(x,y,z)=(x,y,z)$

b) $T(x,y,z)=(x+2y-2z,-x+y+2z,\ x+y+z)$

(iii) $T(x,y)=(2y-x,\quad y,\quad 3y-3x)$.

1.12 : Rank and Nullity of a linear transformation

To a linear transformation T : U $\rightarrow$ V, we associate two sets called the Range space and the null space .

Definition : Let T : U $\rightarrow$ V be a linear transformation .then the range of T is the set of all images of the elements of U under T, and is denoted by R (T) i.e. R (T) = { T(α) : $\alpha \in$ U }, R (T) is also called the range space. Clearly R(T) $\subseteq$ V.

Definition : Let $T : U \to V$ be a linear transformation. Then the kernel of T is the set of all elements of U whose images under T are $0'$ the zero elements of V, and is denoted by N(T).N(T) is also called the null space.

i.e, N(T) is non-empty $\because 0 \in U : T(0) = 0'$ and $N(T) \subseteq U$.

Theorem 1: If $T : U \to V$ is a linear transformation, then R(T) is a subspace of V and N(T) is a subspace of U.

***Proof* :** (i) To prove that R(T) is a subspace of V.

Let $v_1, v_2 \in R(T)$

$\therefore$ there exist $u_1, u_2 \in U$ such that

$T(u_1) = v_1$ and $T(u_2) = v_2$.

Now , $v_1 + v_2 = T(u_1) + T(u_2) = T(u_1+u_2)$ $\because$ T is linear .

$= T(u)$ where $u = u_1+u_2 \in U$.

$\therefore$ there exist some vector $u \in U$ such that

$T(u) = v_1 + v_2$

$\therefore$ $v_1 + v_2 \in R(T)$

Let k be any scalar.

Then $k v_1 \in V$ since V is a vector space .

$\therefore$ $k v_1 = k T(u_1)$ since $v_1 \in R(T)$.

$= T(ku_1)$ since T is a linear.

$\therefore$ there exists an elements $ku_1 \in U$ such that $k v_1 = k T(u_1)$

$\therefore$ $k v_1 \in R(T)$

$\therefore$ R(T) is closed under addition and scalar multiplication.

Hence R(T) is a subspace of V.

(ii) Let $u_1, u_2 \in N(T)$

$\therefore$ $T(u_1) = 0'$ and $T(u_2) = 0'$

Now $T(u_1+u_2) = T(u_1) + T(u_2)$ $\because$ T is linear .

$= 0' + 0' = 0'$

$\therefore$ $u_1+u_2 \in N(T)$

Let c be a scalar .

$T(cu_1) = cT(u_1)$ $\because$ T is linear .

$= c.\ 0' = 0'$

$\therefore$ $c u_1 \in N(T)$

$\therefore$ N(T) is closed w.r.t addition and scalar multiplication.

$\therefore$ N(T) is a subspace of U.

Theorem 2 : Let $T:U \to V$ be a linear transformation . Then T is one-one if and only if N(T) = {0} where 0 is the zero elements of U

***Proof* :** I part : Let T be one-one .

$\therefore$ $T(\alpha_1) = T(\alpha_2) \Rightarrow \alpha_1 = \alpha_2, \forall\ \alpha_1, \alpha_2 \in U$

Let $\alpha \in N(T)$

$\therefore T(\alpha) = 0'$

But $T(0) = 0'$

$\therefore T(\alpha) = T(0)$

This $\Rightarrow \alpha = 0$ as T is one-one.

$\therefore$ we have proved that it $\alpha \in N(T)$ then $\alpha = 0$.

$\therefore N(T) = \{0\}$

II part : let $N(T) = \{0\}$

$T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = 0'$

$\Rightarrow T(\alpha_1) + (-T(\alpha_2)) = 0'$

$\Rightarrow T(\alpha_1) + T(-\alpha_2) = 0'$ since $-T(\alpha_2) = T(-\alpha_2)$.

$\Rightarrow T(\alpha_1 - \alpha_2) = 0'$ since T is linear

$\Rightarrow \alpha_1 - \alpha_2 \in N(T)$.

But $N(T) = \{0\}$ consisting of only elements 0

$\therefore \alpha_1 - \alpha_2 = 0$

$\therefore \alpha_1 = \alpha_2$

Hence $T(\alpha_1) = T(\alpha_2) \Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one .

Definition : If $T: U \to V$ is a linear transformation from a vector space U into another vector space V, the dimension of the range

space R (T) is called the rank of T and is denoted by R (T) and the dimension of the null space (or the kernel) of T is called the nullity of T and is denoted by n (T).

Theorem 3 : Let $T: U \to V$ be a linear transformation .If the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ generates U then the vectors. T (α_1) ,T $(\alpha_2), \ldots T(\alpha_n)$ generates R (T).

Proof **:** Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

Since S spans U, every vector $\alpha \in U$ can be expressed as a linear combination of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$.

Now $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n) \in R(T)$

Since R (T) is a subspace , any linear combination of these vectors is also in R(T)

Let $\beta \in R(T)$. This implies that there exists an $\alpha \in U$ such that $T(\alpha) = \beta$.

Since $\alpha \in U$, $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$.

Since $\beta \in R(T)$, $\beta = T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n)$

$= c_1T(\alpha_1) + c_2T(\alpha_2) + \ldots + c_nT(\alpha_n)$

since T is linear .

$\therefore \beta \in R(T) \Rightarrow \beta =$ a linear combination of $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$

ie, $\beta \in R(T) \Rightarrow \beta$ is in the linear span of $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$

$\therefore$ R (T) is in the span of $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$.

$\therefore$ T $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$ generates R(T)

Remark : From the above theorem, to find the range and rank of T, find the matrix A of the linear transformation and reduce it to echelon form E.

Then (a) The basis of R (T) is the set of non-zero rows of E.

(b) The rank of T = dimension of R(T)

= number of non-zero rows of E.

Theorem 4 : Rank - nullity Theorem :
Let $T : U \rightarrow V$ be a linear transformation and U be a finite dimensional vector space . Then

$$\dim R(T) + \dim N(T) = \dim U.$$

i.e., $r(T) + n(T) = \dim U$. Or rank + nullity = dim (domain)

***Proof* :** Let U be a vector space of dimension m.
i. e., dim U = m.

Since N (T) is a subspace of the finite dimensional vector space U, dimension of N (T) is also finite.

Let dim [N(T)] = n ie, nullity n (T) = n.

Since N(T) is a subspace of U, $n \leq m$.

Let $\beta_1 = \{ \alpha_1, \alpha_2, \ldots \ldots \alpha_n\}$ be a basis of N (T)

$\therefore \beta_1$ is linearly independent in U.

We shall extend this set β_1 to a basis of the vector space U.

Let this basis of U be

$\beta_2 = \{\alpha_1, \alpha_2, \ldots \ldots \alpha_n, \beta_1, \beta_2, \ldots \ldots \beta_s\}$

Clearly n + s = m .

Now $T(\alpha_1), T(\alpha_2), \ldots \ldots T(\alpha_n)$, $T(\beta_1)$, $T(\beta_2), \ldots \ldots T(\beta_s) \in R(T)$

But $T(\alpha_1) = 0'$ $T(\alpha_2) = 0' \ldots$ $T(\alpha_n) = 0'$

since $\alpha_1, \alpha_2, \ldots \ldots \alpha_n, \in N(T)$

Let $S = \{ T(\beta_1), T(\beta_2), \ldots \ldots T(\beta_s)\}$

We shall show that this set S of s vector is a basis of R (T).

(i) S spans R (T)

Since β_2 is a basis of U, it spans U.

Hence the set $\{ T(\alpha_1), T(\alpha_2), \ldots \ldots T(\alpha_n)$, $T(\beta_1)$, $T(\beta_2), \ldots \ldots T(\beta_s)\}$ spans R (T)

Since $T(\alpha_1) = 0$, $T(\alpha_2) = 0, \ldots$ $T(\alpha_n) = 0$

$\therefore$ the set $S = \{ T(\beta_1), T(\beta_2), \ldots \ldots T(\beta_s)\}$ spans R(T).

(ii) S is linearly independent.

Consider $c_1 T(\beta_1) + c_2 T(\beta_2), \ldots \ldots + c_s T(\beta_s) = 0$

$\Rightarrow T(c_1\beta_1 + c_2\beta_2, \ldots \ldots + c_s\beta_s) = 0 \because$ T is linear,

$\Rightarrow c_1\beta_1 + c_2\beta_2, \ldots \ldots + c_s\beta_s \in N(T)$

$\therefore c_1\beta_1 + c_2\beta_2, \ldots \ldots + c_s\beta_s$ can be expressed as a linear combination of the elements of the basis B_1 of N (T).

$\therefore c_1\beta_1 + c_2\beta_2, \ldots \ldots + c_s\beta_s = d_1\alpha_1 + d_2\alpha_2,$
$+ \ldots \ldots + d_n\alpha_n.$

$\Rightarrow d_1\alpha_1 + d_2\alpha_2 + \ldots \ldots + d_n\alpha_n - C_1\beta_1 - C_2\beta_2 - \ldots \ldots - C_s\beta_s = 0$

Since B_2 is a basis of U it is linearly independent.

$\therefore d_1 = 0, d_2 = 0, \ldots \ldots d_n\ 0, c_1 = 0, c_2 = 0, \ldots \ldots c_s = 0$

$\therefore c_1 T(\beta_1) + c_2 T(\beta_2), \ldots \ldots + c_s T(\beta_s) = 0$

$\Rightarrow c_1 = 0, c_2 = 0, \ldots c_s = 0$

$\therefore$ S is linearly independent .

$\therefore$ S is a basis of R (T)

$\therefore$ dim [R(T)]= s

Hence from (1) we get

dim [N(T) + dim [R(T)] = dim U

i.e., n(T) + r(T) = m

or rank + nullity = dim (domain)

Worked Examples :

(1) Find the range space, kernel, rank and nullity of the following linear transformation .Also verify the rank-nullity theorem.
$T : V_2(R) \rightarrow V_2(R)$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

***Solution* :** we shall find the matrix of T w.r.t. the standard

basis { (1,0,),(0,1)} of V_2 (R)

$T(1,0) = (1 + 0, 1) = (1,1)$

$T(0,1) = (0 + 1, 0) = (1,0)$

$\therefore$ The matrix A of T is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} R_2 - R_1$$

This is in echelon form

There are two non-zero rows.

$\therefore$ rank of T = 2

Hence R(T) is the subspace generated by (1, 1) and (0,-1).

$\therefore$ $R(T) = \{x_1 (1,1) + x_2 (0,-1)\}$

$= \{(x_1,x_1) + (0,- x_2)\}$

$= \{ x_1, x_1 - x_2\}$ for $x_1, x_2 \in R$

ie., the range space = $\{x_1, x_1 - x_2 \} = V_2(R)$

To find N(T)

Let $T (x_1, x_2) = 0$

$\Rightarrow (x_1 + x_2, x_1) = (0,0)$

$\Rightarrow x_1 + x_2 = 0, \; x_1 = 0$

$\Rightarrow x_1 = 0, x_2 = 0.$

$\therefore$ N(T) contains only zero element of V_2 (R)

$\therefore$ N(T) = {(0,0,0)} ,i.e, the null space = {(0,0,0)}

$\therefore$ dim [N(T)] = 0 i.e, nullity = 0.

$\therefore$ Rank + nullity = 2 + 0 = 2 = dim V_2 (R)

Hence the rank – nullity theorem is verified.

2. Verify Rank – nullity theorem for the linear transformation

$T : V_3(R) \rightarrow V_2(R)$ *defined by*

$T(x, y, z) = (y - x, \quad y - z)$

Solution : $e_1, e_2, e_3, \in V_3(R)$

$$T(e_1) = T(1,0,0) = (-1,0) = \alpha_1$$
$$T(e_2) = T(0,1,0) = (1,1) = \alpha_2$$
$$T(e_3) = T(0,0,1) = (0,-1) = \alpha_3$$

Consider $\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

The final matrix is in echelon form. It has two non-zero rows

$\therefore$ dim R (T) = 2

$R(T) = \{(1,0),(0,1)\} \in V_2(R) =$ Range space.

To find nullity : $T(x, y, z) = (0,0)$

$$\Rightarrow (y - x, y - z) = (0,0)$$
$$\Rightarrow y - x = 0 \quad and \quad y - z = 0$$
$$\Rightarrow x = y = z$$

$\therefore$ The null space $N(T) = \{(x, x, x) | x \in R\}$

$\therefore$ Nullity = N (T) = 1.

Thus Rank + Nullity = 2 + 1 = 3 = Dim (V_3 (R))

3. Find the linear transformation $T:V_3(R)\to V_3(R)$ **whose image space (range) is spanned by** $\{(1,0,-1),(1,2,2)\}$

Solution : The L.T can be determined when the images of the Vectors belonging to a basis of V_3 (R) is known.

Let $T(e_1)=T(1,0,0)=(1,0,-1)$

$$T(e_2)=T(0,1,0)=(1,2,2)$$

$$T(e_3)=T(0,0,1)=(0,0,0)$$

$$\therefore\ T(x,y,z)=T[xe_1+ye_2+ze_3]$$

$$=xT(e_1)+yT(e_2)+zT(e_3)$$

$$=x(1,0,-1)+y(1,2,2)+z(0,0,0)$$

$$=(x+y,2y,-x+2y)-(1)$$

(1) is the required transformation.

4. Find the linear transformation $T:V_3(R)\to V_3(R)$ **whose image space is spanned by** $\{(1,1,0),(0,1,1)(1,2,1)\}$

Solution : Let $T(e_1)=T(1,0,0)=(1,1,0)$

$$T(e_2)=T(0,1,0)=(0,1,1)$$

$$T(e_3)=T(0,0,1)=(1,2,1)$$

$$\therefore\ T(x,y,z)\ =T[xe_1+ye_2+ze_3]$$

$$=xT(e_1)+yT(e_2)+zT(e_3)$$

$$=x[1,1,0]+y(0,1,1)+z(1,2,1)$$

$$=(x+z,\quad x+y+2z,\quad y+z)$$

5. Find a L.T $V_3(R)\to V_3(R)$ **whose kernel is spanned by** $\{(1,1,1),(1,2,2)\}$

Solution : $T(1,1,-1)=(0,0,0)=T(1,2,2)$

$\therefore$ consider $s=\{(1,1,-1)(1,2,2),(1,0,0)(0,1,0)(0,0,1)\}$

span $V_3(R)$ but linearly dependent.

Let $(x,y,z)=c_1(1,1,-1)+c_2(1,2,2)+c_3(1,0,0)$

$$x=c_1+c_2+c_3,\quad y=c_1+2c_2,\quad z=-c_1+2c_2+0$$

from y & z , $4c_2=y+z\Rightarrow c_2=\dfrac{y+z}{4}$

$$c_2=y-2c_2=y-\left(\frac{y+z}{2}\right)=\frac{y-z}{2}$$

$$\therefore\ c_3=x-c_1-c_2$$

$$=x-\left(\frac{y-z}{2}\right)-\left(\frac{y+z}{4}\right)$$

$$=\frac{4x-2y+2z-y-z}{4}=\frac{4x-3y+z}{4}$$

$$\therefore T(x,y,z)=c_1T(1,1,-1)+c_2T(1,2,2)+c_3T(1,0,0)$$

$$=\left(\frac{y-z}{2}\right)T(1,1,-1)+\frac{y+z}{4}T(1,2,2)+\frac{4x-3y+z}{4}T(1,0,0)$$

$$=\left(\frac{y-z}{2}\right)T(1,1,-1)+\frac{y+z}{4}T(1,2,2)+\frac{4x-3y+z}{4}T(1,0,0)$$

$$=\left(\frac{y-z}{2}\right)(0,0,0)+\frac{y+z}{4}(0,0,0)+\frac{4x-3y+z}{4}(0,0,1)$$

$$=\left(0,\quad 0,\quad \frac{4x-3y+zx}{4}\right)\quad -(1)$$

(1) is the required transformation.

6. T : $R^3 \to R^3$ defined by T (x, y, z) = (x + y, x – y, 2x + z) . Find the range space, null space ,rank and nullity of T and verify rank of T + nullity of T = dim(R^3).

***Solution* :** Let us find the matrix of A of the linear transformation w.r.t. the standard basis{ (1, 0, 0), (0, 1,0), (0,01)}

T (1, 0, 0) = (1 +0,1- 0, 2 + 0) = (1, 1, 2)

T (0, 1, 0) = (0 + 1, 0-1, 0 + 0) = (1, -1, 0)

T (0, 0, 1) = (0 + 0, 0 - 0, 0 + 1) = (0, 0, 1)

$\therefore$ The matrix A of T is A = $\begin{bmatrix}1 & 1 & 2\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$

$$\sim\begin{bmatrix}1 & 1 & 2\\ 0 & -2 & -2\\ 0 & 0 & 1\end{bmatrix}\ (R_2 - R_1)$$

This is in the echelon form and there are three non-zero rows .

$\therefore$ dim [R(T)] = 3 .ie, rank of T= 3

$\therefore$ R(T) = the subspace generated by (1,1,2),(0,-2,-2),(0,0,1)

= x_1 (1,1,2) + x_2 (0,-2,-2) + x_3(0,0,1)

= (x_1,x_1-2x_2,2x_1- 2x_2+x_3) : $x_1,x_2,x_3 \in R$

ie, the range space = {(x_1,x_1-2x_2,2x_1- 2x_2+x_3) : $x_1,x_2,x_3 \in R$}

= R^3.

To find N(T)

Consider T(x_1, x_2, x_3) = ($x_1 + x_2$, $x_1 - x_2$, $2x_1 + x_3$) = (0, 0, 0)

$\Rightarrow$ $x_1 + x_2 = 0$, $x_1 - x_2 = 0$ $2x_1 + x_3 = 0$

$\Rightarrow$ $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

$\therefore$ T (x_1, x_2, x_3) = {(0, 0, 0)} consisting of only zero elements.

$\therefore$ dim [N(T)] = 0 ie, nullity = 0

and N(T) = {(00,0)}ie., the null space = {(0,0,0)}.

Rank + nullity = 3 + 0 =3= dim [R^3]

Hence the rank – nullity theorem is verified

7. T: $V_3(R) \to V_4(R)$ is defined by

$T(e_1) = (0,1,0,2)$, $T(e_2) = (0,1,1,0)$, $T(e_3) = (0,1,-1,4)$

Find range space , null space ,rank and nullity of T and verify the rank nullity theorem. **(M 2002)**

Solution : The matrix A of L.T is

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix} \; R_2\text{-}R_1 \text{ and } R_3 - R_1.$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; (R_3 + R_2)$$

This is in the echelon form.

There are two non- zero rows in this :

Hence dim [R(T)] = 2 i.e., rank of T = 2.

R(T) = The subspace generated by (0,1,0,2) and (0,0,1,-2).

$= \{x_1 (0, 1, 0, 2) + x_2 (0, 0, 1, -2x) : x_1,x_2 \in R\}$

$= \{ (0, x_1, x_2, 2x_1 - 2x_2) : x_1, x_2 \in R\}$

To find N(T)

$T(x_1, x_2, x_3) = 0$

$\Rightarrow T[x_1 (1, 0, 0) + x_2 (0,1,0) + x_3 (0, 0, 1)] = (0,0,0,0)$

$\Rightarrow x_1 T (1,0,0) + x_2 T (0,1,0) + x_3 T (0,0,1) = (0,0,0,0)$

$\Rightarrow x_1 (0,1,0,2) + x_2 (0,1,1,0) + x_3 (0,1,-1,4) = (0,0,0,0)$

$\Rightarrow (0, x_1+ x_2 + x_3 , x_2 - x_3 , 2x_1 + 4x_3) = (0,0,0,0)$

$\Rightarrow x_1+ x_2 + x_3 = 0, x_2 - x_3 = 0, 2x_1 + 4x_3 = 0.$

$\Rightarrow x_1 = -2x_3, x_2 = x_3,x_3 = x_3.$

$\therefore$ $N(T) = \{ (-2x_3 , x_3, x_3) : x_3 \in R\}.$

$\therefore$ dim [N(T)] = 1 i.e, nullity of T = 1.

$\therefore$ rank of T + nullity of T = 2+1 = 3 = dim [V_3 (R)].

Hence the rank – nullity theorem is verified .

8. T:V_3 (R) $\to V_2$ (R) is defined by

$T(e_1) = (2,1)$; $T(e_2) = (0,1)$; $T(e_3) = (1,1)$

Find the range space , kernel ,rank and nullity of T and verify Rank + nullity = dim (domain)

***Solution* :** The matrix of T w.r.t. $\{e_1,e_2,e_3\}$ is

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} R_3 \leftrightarrow 2R_1 \qquad \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} R_3 + R_2.$$

This is in echelon form and there are 2non-zero rows in it .

$\therefore$ dim [R(T)] = 2 i.e, rank of T = 2.

$\therefore$ Range space = { $x_1(1.1) + x_2 (0.1)$}

= { $(x_1,x_1+ x_2)$: $x_1,x_2 \in R$}.

To find kernel,

$T (x_1,x_2,x_3) = T [x_1e_1+ x_2e_2 + x_3e_3]$

$= x_1 T(e_1)+ x_2 T(e_2) + x_3 T(e_3)$

$= x_1 (2,1) + x_2 (0,1) + x_3 (1,1)$

$= (2x_1 + x_3 , x_1 + x_2 + x_3)$

$T (x_1,x_2,x_3) = 0 \Rightarrow (2x_1 + x_3, x_1 + x_2 + x_3) = (0,0)$

$\Rightarrow 2x_1 + x_3 = 0, x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 = x_1, x_2 = x_1 , x_3 = -2x_1$

$\therefore T (x_1,x_2,x_3) = 0 \Rightarrow (x_1,x_2,x_3) =(x_1, x_1, - 2 x_1)$

$\therefore$ N(T) = { $(x_1, x_1, - 2 x_1) : x_1 \in R$}.

in particular ,if $x_1 = 1$,N (T) = {(1,1,-2)}

$\therefore$ dim N (T) = 1 i.e, nullity of T = 1

$\therefore$ rank + nullity = 2 + 1 = 3 = dim (domain)

Hence the rank –nullity theorem is verified.

9. Determine the linear transformation $T:V_3$ (R) $\to V_2$ (R) whose images are generated by the vectors (0,1),(1,1).

Solution **:** consider the standard basis {(1,0,0),(0,1,0),(0,0,1)} of V_3 (R)

Defines T (1,0,0) = (0,1)

T (0,1,0) = (!,1)

T (0,0,1) = (0,0)

T(x,y,z) = T [x(1,0,0) + y (0,1,0) + z (0,0,1)]

= x T (1,0,0) + y T (0,1,0) + z T (0,0,1)

= x (0,1) + y (1,1) + z (0,0)

= (0+ y + 0, x + y + 0)

ie, T(x,y,z) = (y,x + y)

10. Find the linear transformation $T:R^4 \to R^3$ whose null space is generated by (1,2,0,-4) , (2,0,-1,-3).

Solution **:** Define T: $R^4 \to R^3$ such that

T (1, 2, 0, -4) = (0, 0, 0) and T (2, 0, -1, -3) = (0, 0, 0)

Consider the basis of R^4 with (1, 2, 0, -4) and (2, 0, -1, -3)

As two vectors and including (1, 0, 0, 0), (0,1,0,0),(0,0,0,1) and (0,0,0,1)to them .

i. e, S = {(1,2,0,-4) , (2,0,-1,-3), (1,0,0,0),(0,1,0,0),(0,0,0,1) , (0,0,0,1)} spans R^4 but linearly dependent .

To make it linearly independent , delet those vectors in S which can be expressed as linear combination of the preceding ones, so that we get the required basis

Consider (2,0,-1,-3) = a (1,2,0,-4)

$\Rightarrow$ (2,0,-1,-3) = (a,2a,0,-4a)

$\Rightarrow$ a = 2 ,2a = 0 ,0 = -1,-4a = -3 which is impossible

Consider (1,0,0,0) = a (1, 2, 0, -4) + b (2, 0, -1,-3)

= (a + 2b, 2a, -b, -4a - 3b)

$\Rightarrow$ a + 2b = 1, 2a = 0,-b = 0, -4a –3b = 0.

$\Rightarrow$ a = 1,a = 0,b = 0, which is impossible.

Consider (0,1,0,0) = a (1,2,0,-4) + b (2,0,-1,-3) + c (1,0,0,0)

$\Rightarrow$(0,1,0,0) = (a + 2b + c, 2a,-b, -4a –3b)

$\Rightarrow$ a + 2b + c = 0, 2a = 1,-b = 0,-4a –3b = 0.

$\Rightarrow$ a = $\frac{1}{2}$, b = 0, a = 0 which is impossible

Consider (0,0,1,0) = a (1,2,0,-4) + b (2,0,-1,-3) + c (1,0,0,0) + d (0,1,0,0)

$\Rightarrow$ (0,0,1,0) = (a + 2b + c, 2a + d, -b,-4a-3b)

$\Rightarrow$ a + 2b + c = 0, 2a + d = 0,-b = 1,-4a –3b = 0

$\Rightarrow$ a = $\frac{3}{4}$,b = -1, c = $\frac{5}{4}, d = \frac{-3}{2}$.

$\therefore$ $(0,0,1,0)$ is expressed as a linear combination of its preceding ones.

Hence $(1,2,0,-4)(2,0,-1,-3)(1,0,0,0)(0,1,0,0)$ is a basis of R^4

For this basis $T: R^4 \rightarrow R^3$ is defined as

$T(1,2,0,-4)=(0,0,0)T(2,0,-1,-3)=(0,0,0)$

$T(1,0,0,0)=(1,0,0)T(0,1,0,0)=(0,1,0)$

$\therefore$ T is linear.

$T(x_1,x_2,x_{3,}x_4)=(0,0,0)$

$\Rightarrow x_1T(1,2,0,-4)+x_2T(2,0,-1,-3)+x_3T(1,0,0,0)$
$+x_4T(0,1,0,0)=(0,0,0)$

$\Rightarrow x_1(0,0,0)+x_2(0,0,0)+x_3(1,0,0)+x_4(0,1,0)=(0,0,0)$

$\Rightarrow (x_3,x_4,0)=(0,0,0)$

$\Rightarrow x_3=0, x_4=0$

$\therefore (x_1,x_2,x_{3,}x_4)=(x_1,x_2,0,0)$

$=x_1(1,0,0,0)+x_2(0,1,0,0)$

$\therefore$ N (T) is spanned by $\{(1,0,0,0),(0,1,0,0)\}$ which is L.I

$\therefore$ nullity of T = 2.

But $(1,2,0,-4),(2,0,-1,-3)$ also belong to N (T) and are L.I.

Hence they form a basis of N (T)

$\therefore$ N (T) is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$

1.13 Singular and non-singular linear transformation.

Definition : Let U and V be two vector spaces over the same field F. A linear transformation $T:U\rightarrow V$ is said to be singular, If there exists a non-zero vector α such that T (α) $=0'$ and $T:U\rightarrow V$ is said to be non-zero vector of V.

Theorem 1 : A linear transformation $T:U\rightarrow V$ of vector spaces U and V over the same field F, is non-singular if and only if T maps every linearly independent subset of U onto a Linearly independent subset of V

***Proof* :** (i) Let T be non-singular.

Let S = $\{\alpha_1,\alpha_2\cdots\cdots\cdots\alpha\}$ be a linearly independent subset of U.

We shall show that $\{T(\alpha_1),T(\alpha_2)\cdots\cdots\cdots T(\alpha_n)\}$ is linearly independent.

Consider $a_1T(\alpha_1)+a_2T(\alpha_2)+\cdots\cdots\cdots+a_nT(\alpha_n)=0'$

$\Rightarrow T\left(a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots\cdots + a_n\alpha_n\right) = 0' \because$ T is linear.

$\Rightarrow Ta_1\alpha_1 + a_2\alpha_2 + \cdots\cdots\cdots + a_n\alpha_n = 0' \because$ T is non-singular

$\Rightarrow a_1 = 0, a_2 = 0 \cdots\cdots\cdots a_n = 0' \because$ S is L.I.

$\therefore S'$ is linearly independent.

(ii) Conversely, let T map every linearly independent subset of U onto a linearly independent subset of V.

If α is a non-zero vector of U, then $\{\alpha\}$ is L.I and hence by hypothesis $\{T(\alpha)\}$ is L.I.

Consequently, T $(\alpha) \neq 0'$

Hence $T(\alpha) = 0' \Rightarrow \alpha = 0$

$\therefore$ T is non-singular.

Theorem 2: A linear transformation $T: U \rightarrow V$ is an Isomorphism if and only if T is non-singular.

***Proof* :** (i) Let T be an isomorphism

$\therefore$ T is one-one.

Let $\alpha \in U$ and $T(\alpha) = 0'$

But $T(0) = 0'$

$\therefore T(\alpha) = T(0) \Rightarrow \alpha = 0$

$\therefore T(\alpha) = 0' \Rightarrow \alpha = 0.$

$\therefore$ T is non-singular.

(ii) Conversely, let T be non-singular.

$\Rightarrow T(\alpha) - T(\beta) = 0$

$\Rightarrow T(\alpha - \beta) = 0 \quad \because T$ is linear

$\Rightarrow \alpha - \beta = 0 \quad \because T$ is non-singular

$\Rightarrow \alpha = \beta$.

$\therefore$ T is one-one.

Cor I : A linear transformation on a finite dimensional vector space is onto if and only if T is non-singular

Proof : T is invertible iff T is one-one and onto and T is one-one and onto if and only if T is non-singular.

Cor 2 : Let U and V be two finite dimensional vector speaces ofve the same field F and let T be a linear transformation from U onto V. Then dim U=dim V if and only if T is non-singular.

***Proof* :** Since dim U = dim[R(T)] + dim[N(T)]

$\therefore$ dim U = dim V $\Leftrightarrow$ dim U = dim [R(T)] $\because$ R(T) = V.

$\Leftrightarrow$ dim[N(T)] = 0

$\Leftrightarrow$ dim N(T) = 0

$\Leftrightarrow$ T is non-singular.

Worked Examples

1. Give an example of a linear map which is one – one but not onto

Let P(t) be a vector space of polynomials over the field of Reals.

Define $T : p(t) \rightarrow p(t)$ by $T(\alpha) = t\alpha \;\; \forall \alpha \in p(t)$

T is one –one $\because \quad T(\alpha) = t\alpha \quad \forall \alpha \in p(t)$

$\Rightarrow t\alpha = t\beta$

$\Rightarrow \alpha = \beta$

T is not onto $\because$ there exist no polynomial α such

$T(\alpha) = p(t)$

2. Show that $T = R^2 \rightarrow R^2$ defined by $T(x, y) = (x - y, x - 2y)$

is non – singular and find its inverse.

Solution : Given $T(x,y)=(x-y,x-2y)$

If $(x-y, \quad x-2y)=(0,0)$

$\Rightarrow x-y=0, \quad x-2y=0$

$\Rightarrow x=y=0$

$\therefore$ T is non – singular

Let $T^{-1}(x_1,x_2)=(y_1,y_2)$

$\Rightarrow (x_1,x_2)=T(y_1,y_2)$

$\Rightarrow (x_1,x_2)=(y_1-y_2, \quad y_1-2y_2)$

$\Rightarrow y_1-y_2=x_1 \quad , \quad y_1-2y_2=x_2$

$\Rightarrow y_2=x_1-x_2 \quad , \quad y_1=2x_1-x_2$

$\therefore T^{-1}(x_1, x_2)=(2x_1-x_2, \; x_1-x_2) \qquad -(1)$

(1) is the required inverse

3. Give an example of a linear map which is onto but not one – one.

Solution : Let $p(t)$: set of polynomials over the field R

Define $T:p(t)\to p(t)$ by $T(\alpha)=\dfrac{d\alpha}{dt} \quad \forall \quad \alpha\in p(t)$

It is onto but not one – one because

$T(2x^2+4x+7)=T(2x^2+4x-7)$

$\Rightarrow 2x^2+4x+7 \neq 2x^2+4x-7$

EXERCISE

1. Show that each of the following L.T is non – singular and find its inverse

a) $T:R^3\to R^3$ defined by $T(x,y,z)=(x+y+z, \; y+z, \; z)$

b) $T:R^3\to R^3$ defined by $T(x,y,z)=(x+z, x+y+z, y+z)$

c) $T:R^3\to R^3$ defined by

$T(x,y,z)=(x+y-2z, x+2y+z, 2x+2y-3z)$

Answers

1. a) T^{-1} = (x – y, y – z, z) b) T^{-1} = (y – z, y – x, x – y + z)
   c) T^{-1} = (- 8x – y + 5z, 5x + y – 3z, z – 2x)

1.14 Eigen Values and Eigen Vectors of a linear transformation

Definition : Let A be a square matrix over a field. F. The matrix A - λI where I is the unit matrix of the same order as that of A and λ is an indeterminat, is called the *Characteristic matrix of A.*

Definition : If A is a square matrix of order n x n then the determinant $|A-\lambda I|$ is a non-zero polynomial of degree n in λ. This polynomial is called the Characteristic polynomial of A.

Definition : The equation $|A-\lambda I|$ = 0 is called the Characteristic equation of A or Eigen equation of A.

Definition : The roots of the characteristic equation $|A-\lambda I|$ = 0 are called the Characteristic roots or Eigen Values of A.

If $A = \begin{vmatrix} a_{11} & a_{12}\text{.........} & a_{1n} \\ a_{21} & a_{22}\text{.........} & a_{2n} \\ \text{............................} & & \\ \text{............................} & & \\ a_{n1} & a_{n2}\text{.........} & a_{nn} \end{vmatrix}$

Then $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12}\text{.........} & a_{1n} \\ a_{21} & a_{22}-\lambda\text{.........} & a_{2n} \\ \text{............................} & & \\ \text{............................} & & \\ a_{n1} & a_{n2}\text{.........} & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\text{................}(a_{nn}-\lambda) +$ terms with atmost (n-2) factors of the form $a_{ii}-\lambda$

$\therefore c(\lambda) = (-1)^n[\lambda^n + c_{n-1}\lambda^{n-1} + c_{n-2}\lambda^{n-2} + \text{.........} + c_1\lambda + c_0] = 0$

where $c_{n-1}, c_{n-2}\text{....................}c_1, c_0$ are constant.

Worked Examples :

(1) Find the eigen values of the matrices

(i) $\begin{bmatrix} 7 & 6 \\ 5 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Solution : (i) Let A = $\begin{bmatrix} 7 & 6 \\ 5 & 8 \end{bmatrix}$

$$\therefore A-\lambda I = \begin{bmatrix} 7-\lambda & 6 \\ 5 & 8-\lambda \end{bmatrix}$$

$$\therefore |A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 7-\lambda & 6 \\ 5 & 8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)(8-\lambda) - 30 = 0$$
$$\Rightarrow \lambda^2 - 15\lambda + 26 = 0$$
$$\Rightarrow (\lambda-2)(\lambda-13) = 0$$
$$\Rightarrow \lambda = 2, \lambda = 13$$

(ii) Let A = $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

$$\therefore A-\lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

$$|A-\lambda I| \Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2] - 1[2-2(2-\lambda)] = 0$$
$$\Rightarrow (1-\lambda)(\lambda^2-5\lambda+4) - (-2+2\lambda) = 0$$
$$\Rightarrow \lambda^2 - 5\lambda + 4 - \lambda^3 + 5\lambda^2 - 4\lambda + 2 - 2\lambda = 0$$
$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$
$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\Rightarrow \lambda = 1, 2, 3.$$

$\therefore$ the Eigen values are 1,2,3.

(2) If A is any square matrix, then prove that A and A^T have the same eigen values.

***Solution* :** Consider $(A-\lambda I)^T = A^T - (\lambda I)^T$

$$= A^T - \lambda I$$

But $|A-\lambda I| = |(A-\lambda I)^T| = |A^T - \lambda I|$

$\therefore$ A and A^T have the same characteristic (or Eigen) equation and hence the same Eigen values.

Definition : Let $T:V \to V$ be a linear transformation of an n dimensional vector space V, and A be the matrix of the linear transformation T. Then the characteristic equation (or Eigen equation) of T is defined as the characteristic equation of A i.e., $|A-\lambda I|=0$. The roots of the characteristic equation (or the equation) $|A-\lambda I|=0$ are called the *characteristic roots of the Eigen values* of T.

Definition : If $T:V \to V$ is a linear transformation of an dimensional vector space V, A is an n × n matrix of T, and λ is an Eigen value of T, then the vector $x=(x_1, x_2 \cdots\cdots x_n)$ which satisfies the equation $Ax=\lambda x$ is called the Eigen vector corresponding to the value of λ.

The vector $x=(x_1, x_2 \cdots\cdots x_n)$ can be represented as the column *matrix* $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

The equation $Ax=\lambda x$ for the values of $\lambda=\lambda_1, \lambda_2 \cdots\cdots \lambda_n$ in the
Matrix for is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ . \\ . \\ . \\ \lambda x_n \end{bmatrix}$$

n × n

The set of all vectors $x \in R^n$ which satisfy the equation

$Ax=\lambda x$ fro a given λ forms a subspace of R^n called the Eigen space of A corresponding to λ.

Working rule to find the Eigen vectors of a L.T.

(i) Find the matrix A of the linear transformation $T:V \to V$.

(ii) Find the Eigen equation of A i.e., $|A-\lambda I|=0$

(iii) Find the Eigen values $\lambda=\lambda_1, \lambda_2, \lambda_3 \cdots\cdots$ by solving the equation $|A-\lambda I|=0$

(iv) Then to find the Eigen vector corresponding to $\lambda=\lambda_1$, put $\lambda=\lambda_1$ in $[A-\lambda I]x=0$. we get n equations in n unknowns. The solution of this corresponding to λ_1.

Similarly, determine the Eigen vectors corresponding to $\lambda=\lambda_2, \lambda=\lambda_3$. etc.

Worked Examples :

(1) Find the basis for the Eigen space of the L.T. $T:R^2 \to R^2$ defined by $T(x,y)=(x+y, y)$.

Solution : First, we shall find the matrix of T w.r.t standard basis $\{(1,0), (0,1)\}$

$T(1,0)=(1,0)$

$T(0,1)=(1,1)$

∴ The matrix of the L.T. is $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Eigen equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix}=0$

$$\Rightarrow (1-\lambda)(1-\lambda)-0=0$$

$$\Rightarrow \lambda=1, \lambda=1$$

Let $x=(x_1, x_2)$ be a vector in R^2

Then $Ax=\lambda x$

$$\Rightarrow (A-\lambda I)=0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1-\lambda)x_1 \\ x_1+(1-\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (1-\lambda)x_1=0, x_1+(1-\lambda)x_2=0.$$

Put $\lambda=1$, we get $x_1=0, x_2=0.$

$\therefore$ The Eigen vector is (0, 0) ^{1}

The Eigen space is { (0,0) }

(2) Find the Eigen of the L.T. $T: R^3 \to R^3$ **defined by**

$$T(x,y,z)=(2x+y, y-z, 2y+4z)$$

Solution : $T(1,0,0)=(2,0,0)$

$$T(0,1,0)=(1,1,2)$$

$$T(0,0,1)=(0,-1,4)$$

$\therefore$ The matrix of L.T is $A=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 4 \end{bmatrix}$

$\therefore$ Eigen equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(4-\lambda)+2\right]=0$$

$$\Rightarrow (2-\lambda)\left(\lambda^2-5\lambda+6\right)=0$$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow (2-\lambda)^2(\lambda-3)=0$$

$$\Rightarrow \lambda=2,2,3.$$

$\therefore$ Eigen values are 2,3

Consider $Ax=\lambda x$

i.e., $(A-\lambda I)x=0$

i.e., $\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e., $(2-\lambda)x_1+0x_2+0x_3=0 \quad \Rightarrow x_1=0$

$$x_1+(1-\lambda)x_2+x_3=0 \quad \Rightarrow (1-\lambda)x_2+2x_3=0$$

$$0x_1-x_2+(4-\lambda)x_3=0 \quad \Rightarrow -x_2+(4-\lambda)x_3=0$$

Put $\lambda=2 \;\therefore -x_2+2x_3=0 \Rightarrow x_2=2x_3$

$\therefore If\; x_3=k, x_2=2k$

$\therefore$ the vector is (0,2k,k) $\therefore$ (0,2,1)` is a basis of the

Eigen space corresponding to λ = 2

Put λ = 3, then $-2x_2+2x_3=0 \Rightarrow -x_2+x_3=0$ and

$-x_2+x_3=0 \quad \therefore x_2=x_3=k$

$\therefore$ the vector is (0,k,k). $\therefore$ (0,1,1) is a basis of the Eigen space corresponding to λ = 3.

(3) Find the Eigen values and Eigen vectors of the linear transformation $T : R^3 \to R^3$ **defined by** $T(e_1) = (1,1,0)$, $T(e_2) = (0,1,1)\,T(e_3) = (1,2,1)$

***Solution* :** The matrix of the L.T. is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$\therefore$ The Eigen equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - 2\right] - 1(0-1) + 0 = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2 - 2\lambda - 1\right) + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 1 - \lambda^3 + 2\lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + \lambda = 0$$

$$\Rightarrow \lambda = 0, \lambda = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{52}}{2}$$

$\therefore$ the Eigen values are $\lambda = 0\,, \frac{3 \pm \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}$.

$$Ax = \lambda x \Rightarrow (A - \lambda I)\,x = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (1-\lambda)\,x_1 + x_2 = 0$$

$$(1-\lambda)\,x_2 + 1x_3 = 0$$

$$1x_1 + 2x_2 + (1-\lambda)\,x_3 = 0$$

Put λ = 0. $\therefore$ $x_1 + x_2 = 0$ --(1)

$x_2 + x_3 = 0$ --(2)

$x_1 + 2x_2 + x_3 = 0$ --(3)

(1) is $x_1 + x_2 = 0$. $\therefore x_2 = -x_1$

$\therefore$ from (2) we get $x_3 = -x_2$

i.e., $x_3 = x_1$

$\therefore$ The vector $(x_1, x_2, x_3) = (x_1, -x_1, x_1)$

$\therefore \left\{(1,-1,1)'\right\}$ is a basis of the subspace corresponding to λ=0

Put $\lambda = \dfrac{3+\sqrt{5}}{2}$ $\therefore \left(1-\dfrac{3+\sqrt{5}}{2}\right)x_1 + x_2 = 0$ --(4)

$$\left(1-\frac{3+\sqrt{5}}{2}\right)x_2 + x_3 = 0 \qquad \text{--(5)}$$

$$x_1 + 2x_2 + \left(1-\frac{3+\sqrt{5}}{2}\right)x_3 = 0 \qquad \text{--(6)}$$

From (4) $x_2 = -\left(1-\dfrac{3+\sqrt{5}}{2}\right)x_1 = \left(\dfrac{1+\sqrt{5}}{2}\right)x_1$

From (5) $x_3 = -\left(1-\dfrac{3+\sqrt{5}}{2}\right)x_1 = \left(\dfrac{1+\sqrt{5}}{2}\right)x_1$

$$= \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)x_1$$

i.e., $x_3 = \dfrac{6+2\sqrt{5}}{4}x_1 = \dfrac{3+\sqrt{5}}{2}x_1$

$\therefore$ The vector is $\left[x_1, \left(\dfrac{1+\sqrt{5}}{2}\right)x_1, \left(\dfrac{\sqrt{5}+3}{2}\right)x_1\right]'$

$$= x_1\left(1, \frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}+3}{2}\right)$$

$\therefore \left\{\left(1, \dfrac{1+\sqrt{5}}{2}, \dfrac{\sqrt{5}+3}{2}\right)'\right\}$ is a basis of the subspace.

Put $\lambda = \dfrac{3-\sqrt{5}}{2}$

$\therefore$ $\left(1-\dfrac{3-\sqrt{5}}{2}\right)x_1 + x_2 = 0$ ---(7)

$$\left(1-\frac{3-\sqrt{5}}{2}\right)x_1 + x_2 = 0 \qquad \text{---(8)}$$

$$x_1 + 2x_2 + \left(1-\frac{3-\sqrt{5}}{2}\right)x_3 = 0 \qquad \text{---(9)}$$

$$x_2 = -\left(\frac{-1+\sqrt{5}}{2}\right)x_1 = \left(\frac{1-\sqrt{5}}{2}\right)x_1$$

$$= \left(\frac{6-2\sqrt{5}}{4}\right)x_1 = \left(\frac{3-\sqrt{5}}{2}\right)x_1$$

$\therefore$ The vector $(x_1, x_2, x_3) = \left(x_1, \left(\dfrac{1-\sqrt{5}}{2}\right)x_1, \left(\dfrac{3-\sqrt{5}}{2}\right)x_1\right)'$

$$= x_1\left(1, \frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)'$$

$\therefore \quad \left\{\left(1, \frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)'\right\}$ is a basis of the subspace.

EXERCISE

I 1) Show that $V_3(R) \to V_2(R)$ defined by

(i) $T(x, y, z) = (x+y, y+z)$ is a linear transformation (N 03)

(ii) $T(x, y, z) = (x-y, y-z)$ is a linear transformation

(iii) $T(x, y, z) = (x+y+z, x-y)$ is a linear transformation (N 03)

(iv) $T(x, y, z) = (2x-3y, 3y+4z)$ is a linear transformation

(v) $T(x, y, z) = (y-x, y-z)$ is a linear transformation (095)

2) Prove that $T : R^3 \to R^3$ defined by

(i) $T(x, y, z) = (x+y,\ x-y,\ 2x+z)$ is a L.T (M 93)

3) Prove that $T(x, y) = (x+y,\ x-y,\ y)$ is a linear transformation. (A 98)

II 1) Verify whether the following are linear transformations:

(i) T: $V_2(R) \to V_3(R)$ defined by T(x, y) = (x, y, 0)

(ii) T: $V_2(R) \to V_2(R)$ defined by T(x, y) = (2x, y)

(iii) T: $V_2(R) \to V_2(R)$ defined by T(x, y) = (x^2, y)

(iv) T: $V_2(R) \to V_2(R)$ defined by T(x, y) = (3x+2y, 3x-4y)

(v) T: $V_2(R) \to V_2(R)$ defined by T(x, y) = (x+y, y)

(vi) T: $R^2 \to R^2$ defined by T(x, y) = (2x+y, x-y)

(vii) T: $R^3 \to R^2$ defined by T(x, y,z) = (2x+y, 3y-4z)

(viii) T: $R^3 \to R^1$ defined by T(x, y, z) = 2x-3y+4z

(ix) T: $R^3 \to R^3$ defined by T(x, y,z) = (x,y,z)

(x) T: $R^3 \to R^3$ defined by T(x, y,z) = (x+2y-z, y+z, x+y-2z)

(xi) T: $R^2 \to R^3$ defined by T(x, y) = (x+y, 2y, x+1)

(xii) T: $R^2 \to R^4$ defined by T(x, y) = (x,y,y,y)

(xiii) T: $V_3(R) \to V_2(R)$ defined by T(x, y,z) = (x+z, x+y+z)

(2) Find the linear transformation :

(i) T: $R^2 \to R^1$ dif T(1, 1) =3, T (0, 1) = - 2

(ii) T: $R^2 \to R^2$ if T(1, 1) = (3, 0), T (2, 1) = (1, 2)

(iii) T: $R^2 \to R^4$ if T(1, 1) =(1, 1, 1, 1),

T (1, -1) = (1, -1, -1, -1) (3, 0)

(iv) T: $V_2(R) \to V_3(R)$ if T(1, 2) = (3, -1, 5), T(0, 1) = (2, 1, -1)

(v) T: $R^2 \to R^2$ if T(2, 1) = (3, 4), T(-3, 4) = (0, 5)

(vi) T: $R^2 \to R^2$ if T(1, 1) = (1, -1,1, -1), T(-1, 2) = (-1, -2, -1, -2)

(vii) T: $R^3 \to R^3$ if T(1, 0, 0) = (4, 5, 8),

T(1, -1, 0-) = (8, 10, 18) ; T(0, 1, 1) = (-3, -4, -7)

(viii) T: $R^3 \to R^3$ if T(1, 1, 1) = (1, 1, 1)

T(1, 2, 3) = (-1, -2, -3) ; T(1 ,1, 2) = (2, 2, 4)

(ix) T: $R^2 \to R^3$ if T(1, 0) = (1, 0, 1), T(0, 1) = (-1, 1, 1)

(x) T: $R^3 \to R^3$ if T(1, 1, 1) = (2,1, 1), T(1, 2, 1) = (3, 2, 1) T(1, 0, 0) = (1, 0, 0)

(3) Let M(R) be a vector space of all n x n matrices over R and B be any fixed non-zero matrix of M(R). Show that T : M (R) → M (R) defined by (i) T (A) = AB – BA, (ii) T(A) = BA (iii) T (A) = AB^2 + BA are linear transformations and (iv) T(A) = B + A is not linear unless B is a zero matrix.

(4) (i) Show that T : $R^2 \to R^2$ defined by T(x, y) = (x + 2, y + 3) is not linear.

(ii) Show that T : R→ R defined by T(ab) = ab is not linear

(5) If V is the vector space of all real valued functions defined on (0, 1), then show that T : V→ R^2 defined by T(f) = {f(0), f(1)} is linear.

(6) Find (x, y, z) when L.T is defined by

$$T(1,1,1)=3\ ,\quad T(0,1,-2)=1,\quad T(0,0,0)=-2.$$

(7) Consider the basis $s=\{x_1, x_2, x_3\}$ of R^3 where $x_1=(1,1,1)$ $x_2=(1,1,0)$ $x_3=(1,0,0)$. Express $(2,-3,5)$ in terms of the vectors x_1, x_2, x_3

8) For the following linear transformation, find the range space, null space, rank, nullity and verify the rank – nullity theorem.

(i) T: $V_3(R) \to V_3(R)$ defined by T(x, y, z) = (x+ y, x – y, 2x + z)

(ii) T: $V_3(R) \to V_2(R)$ defined by T(x, y, z) = (y-x, y-z)

(iii) T: $V_3(R) \to V_2(R)$ defined by

$T(e_1) = e_1 + e_2 + e_3$, $T(e_2) = e_1 - e_2 + e_3$

$T(e_3) = e_1 - 3e_2 + 3e_3$

(iv) T : $R^3 \to R^4$ given by T (1, 0, 0) = (0, 1, 0, 2)

T(0, 1, 0) = (0, 1,1, 0), T(0, 0, 1) = (0, 1, -1, 4)

(v) T: $R^3 \to R^3$ given by $T(x_1, x_2, x_3) = (x_1, x_3, x_2)$

(vi) T : $R^3 \to R^3$ given by

T(x, y, z) = x + y, x + z, y + z

(vii) T : $R^3 \to R^3$ given by $T(e_1) = e_1 - e_2$; $T(e_2) = 2e_1 + e_3$;

$T(e_3) = e_1 + e_2 + e_3$

(viii) $T : R^3 \to R^2$ given by $T(e_1) = (2, 1)$, $T(e_2) = (0, 1)$, $T(e_3) = (1, 1)$

(ix) $T : R^3 \to R^3$ given by

$T(1, 0, 0) = (1, -1, 0)$, $T(0, 1, 0) = (2, 0, 1)$,

$T(0,0, 1) = (1, 1, 1)$

(x) $T : R^3 \to R^2$ given by $T(x, y, z) = (x + y, y + z)$

(9) Find a linear transformation $T : R^3 \to R^4$ whose range space is generated by (1, 2, 0, -4) and (2, 0, -1, -3)

(10) Find a linear transformation $T : R^3 \to R^3$ whose range space is generated by (1, 2, 3) and (4,5,6)

(11) Find the linear transformation $T : R^4 \to R^3$ whose kernel is generated by (1, 2, 3, 4) and (0,1, 1, 1)

(12) Find the linear transformation $T : R^3 \to R^3$ whose null space is generated by (1, 1, -1) and (1, 2, 2)

(13) Find the linear transformation $T : R^3 \to R^3$ whose range space is spanned by {(1, 2, 2) , (1, 0, -1)}

(14) Find the Eigen values and Eigen vectors of the following linear transformations:

(i) $T: V_2(R) \to V_2(R)$ defined by $T(1, 0) = (1, 2)$; $T(0, 1) = (3, 2)$

(ii) $T: V_2(R) \to V_2(R)$ defined by $T(e_1) = (1, 4)$, $T(e_2) = (2, 3)$

(iii) $T: R^3 \to R^3$ defined by $T(e_1) = (4, 0,1)$, $T(e_2) = (-2, 1, 0)$, $T(e_3) = (-2, 0, 1)$

(iv) $T: R^3 \to R^3$ defined by $T(1, 0, 0) = (1, -3, 3)$, $T(0, 1, 0) = (3, -5, 3)$, $T(0, 0, 1) = (6, -6, 4)$

(v) $T: R^3 \to R^3$ defined by $T(1, 0, 0) = (-3, 1, -1)$, $T(0, 1, 0) = (-7, 5, -1)$, $T(0, 0, 1) = (-6, 6, 2)$

(vi) $T: R^3 \to R^3$ given by $T(e_1) = (3, 2, 4)$, $T(e_2) = (2, 0, 2)$, $T(e_3) = (4, 2, 3)$

(vii) $T: R^3 \to R^3$ given by

$T(x, y, z) = (3x + 2y + z, x + 4y + z, x + 2y + 3z)$
(A 97)

(viii) $T : V_3 (R) \to V_3 (R)$ given by

$T(x, y, z) = (x, x + y, z)$

(ix) $T : V_3 (R) \to V_3 (R)$ given by

$T(x, y, z) = (3x, 2y +z, -5y-2z)$

(x) $T: R^3 \to R^3$ given by

$T(x, y, z) = (x + 3z, 2x + y – z, x – y + z)$

Answers

I. 6. $8x-3y-2z$ 7. $5x_1-8x_2+5x_3$

II (1) (iii), (xi) are not linear. The others are linear

(2) (i)$T (x, y) = 5x – 2y$, (ii) $T(x, y) = \left(\frac{5y-x}{3}, \frac{4x-2y}{3}\right)$

(iii) $T(x, y) = (x, y,y, y)$,

(iv) T (x, y) = (-x + 2y , -3x + y, 7x - y)

(v) T (x, y) = $\left(\frac{12x+9y}{11}, x+2y\right)$ (vi) T(x, y) = (x, -y, x, -y)

(vii) T(x, y, z) = (4x – 4y + z, 5x – 5y + z, 8x – 10y + 3z)

(viii) T(x, y, z) = (4x – 4y + z, 5x – 5y + z, 8x – 10y + 3z)

(ix) T(x, y) = (x-y, y, x + y)

(x) T(x, y, z) = (x + y, y, z)

6. 8x – 3y – 2z, (7) $5x_1 - 8x_2 + 5x_3$

8. (i) R(T) = {(x, x + y, 2x + y)}, x , y $\in$ R

N(T) = {(0, 0, 0)} Rank = 3, nullity = 0

(ii) R(T) = subspace spanned by {(1, 0), (0, 1)} = V_2 (R)

rank = 2, nullity = 1

(iii) {(1, 1, 1), (1, -1, 1), (1, -3, 3)} rank =3, nullity = 0

(vi) R(T) = subspace spanned by {(0, 1, 1,0), (0, 1, -1,4)},

N(T) = {(-2, 1, 1)}

(v) R(T) = subspace spanned by {(1, 0, 0), (0, 1,0),

(0, 0, 1)}, N(T) = {(0, 0, 0)} rank = 3, nullity = 0

(vi) R(T) = R_3, N (T) = {(0, 0, 0)} rank = 3, nullity = 0

(vii) R(T) = subspace spanned by {(1, 1, 0), (2, 0, 1)},

N(T) = {(1, 1, -1)} Rank = 2, nullity = 1

(viii) R(T) = subspace spanned by {(0, 1), (1, 1)}

N(T) = {(1, 1, -2)} Rank = 2, nullity = 1

(ix) R(T) = subspace spanned by {(1, 1, 0), (2, 0, 1)},

N(T) = {(1, 1, -1)} Rank = 2, nullity = 1

(x) R(T) = subspace spanned by {(1, 0), (0, 1)}

N(T) = {(1, -1, 1)} Rank = 2, nullity = 1

9. T(x, y, z) = (x + 2y, 2x-y, -4x-3y)

10. T(x, y, z) = (x + 4y, 2x+ 5y, 3x+6y)

11. T(x, y, z) = (x + y –z, 2x + y – t , 0)

12. T(x, y, z) = (0, 0, $\frac{4x-3y+z}{4}$)

13. T(x, y, z) = (2 + y, 2y, 2y - x)

14. (i) λ = 4, -1; (2, 3), (1, -1) (ii) λ = 5, -1; (1,1), (-2, 1)

(iii) λ = 1, 2, 3 ; (0, 1, 0), (1, -2, -2)

(iv) λ = 4, -2; (1, 1,2), (0, 1, 1)

(v) λ = 2, 4, -2 ; (1, 1, -4) ; (1, 1, 0)

(vi) λ = 0, -1, 7 ; (1, 2, -2) (vii) λ = 2, 6 ; (1, 2,- 3), (1, 2,1)

(viii) λ = 1 ; (1, 0, 2) (ix) λ = 3 ; (1, 0, 0)

(x) λ = 2; (7, 6, -15)