LAGRANGIAN AND HAMILTONIAN

A. Constraints and Degrees of Freedom .

A constraint is a restriction on the freedom of motion of a system of particles in the form of a condition. The number of independent ways in which a mechanical system can move without violating any constraints which may be imposed on the system is called the number of degrees of freedom of that system. In other words, number of degrees of freedom is the number of independent variables that should be specified in order to describe the positions and velocities of all the particles in the system which does not violate any condition.

The motion of a free particle can be specified by three independent coordinates such as the Cartesian coordinates x, y and z or spherical polar coordinates r, θ, ϕ and so on. Hence, the free particle has three degrees of freedom. For a particle constrained to move only in a plane , like the particle executing circular motion or a projectile, two independent coordinates x, y or r, θ is sufficient to describe the motion. Hence, the particle has two degrees of freedom. A particle confined to move along a curved path has only one degree of freedom.

When the motion of a system is restricted in some manner, constraints are said to have been introduced. A bead sliding down a wire or a disc rolling down an inclined plane are some examples of constrained motion.

Every condition of constraint reduces the number of degree of freedom by one. Thus, a system having N particles moving independent of one another has 3N degrees of freedom and if their motion is restricted by k conditions of constraints, the true number of degrees of freedom will be only (3N – k).

When the conditions of constraints can be expressed as equations connecting the coordinates of the particles, the constraints are called **holonomic**.

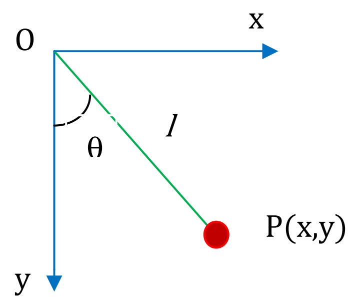
The general constraining equation is of the form

$$f_c\,(r_1, r_2, r_3, \dots, r_n, t) = 0$$

In the case of a simple pendulum moving in the X-Y plane, the two equations of constraints are :

$$x^2 + y^2 = l^2 = \text{const}$$

$$z = 0$$



In the case of the simple pendulum shown, only one variable q is sufficient to locate the position of the oscillating bob P.

When the conditions of constraints cannot be expressed as an equation but may be expressed as an inequality, then the constraints are called non-holonomic.

The equation of constraint in the case of a particle moving on or outside the surface of a sphere of radius a is $x^2 + y^2 + z^2 \geq a^2$ if the origin of the coordinate system coincides with the centre of the sphere. This inequality is a non-holonomic constraint.

B. *Scleronomous and Rheonomous constraints.*

Scleronomous constraints are those which are independent of time. Rheonomous constraints are those which are explicitly dependent on time.

The oscillations of the bob of a simple pendulum whose length is constant is scleronomous whereas whose length varies with time, the constraint is rheonomous.

C. Generalised coordinates.

Consider a system of N particles where the motion is restricted by k conditions of constraints expressed as k equations connecting the 3N coordinates. The position coordinates of particles expressed as (x_1, y_1, z_1), (x_2, y_2, z_2),.... can now be relabelled as (x_1, x_2, x_3), (x_4, x_5, x_6),.... Thus, the coordinates of the N particles will now run as $(x_1, x_2 x_3, ...,x_{3N})$

Instead of (x_1, y_1, z_1),... (x_N, y_N, z_N). The dynamics of the system can be understood by solving the 3N equations

$$m_i \frac{d^2 x_i}{dt^2} = F_{i(\, i=1,2,\ldots N)}$$

In this case, $m_1 = m_2 = m_3 =$ mass of first particle

$m_4 = m_5 = m_6 =$ mass of second particle

and so on.

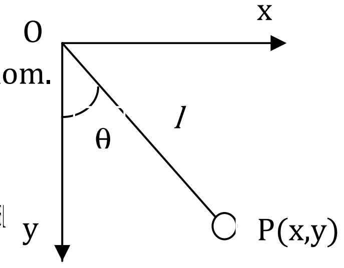
However, these 3N equations are not independent since the conditions of constraints expressible as k equations connecting the coordinates x_i's $\{i.e., f_i(x_1, x_2, x_3, \ldots, x_{3N}) = a_i\}$ must also be taken into account in obtaining consistent solutions.

It is possible that a new set of coordinates $q_1, q_2, q_3, \ldots, q_n$ can be chosen properly and expressed in terms of the original set of coordinates (x_i) such that their number n is equal to the number of degrees of freedom {n=3N-k}.

The equations of motion described in terms of new coordinates (q's) will be independent since the choice of q's is such that they are no longer connected to each other by an equation of constraint. The new coordinates q's are known as generalised coordinates. The generalised coordinates need not have the dimensions of length.

Consider the oscillations of the bob of a simple pendulum of length l oscillating in the X-Y plane to be described by the coordinates x and y with the origin O coinciding with the point of suspension. The coordinates x and y are not independent of one another but are connected by the equation of constraint as

$$x^2 + y^2 = l^2 = \text{const}$$



The bob though moves in a plane has only one degree of freedom.

Its motion can be described in terms of x and y by

the relation $\theta = \tan^{-1}\frac{x}{y}$. The angle θ serves as the generalized coordinate in this case.

Generalised Force.

The generalised coordinates (q 's) may undergo displacement from the initial values ($q_1, q_2, q_3, \ldots, q_n$) to their neighboring values ($q_1+\delta q_1$, $q_2 + \delta q_2$, $q_3+ \delta q_3$,, ... , $q_n+\delta q_n$) over an interval of time δt. The corresponding changes in the original coordinates can be expressed as

$$\delta x_1 = \frac{\partial x_1}{\partial q_1}.\delta q_1 + \frac{\partial x_1}{\partial q_2}.\delta q_2 + \ \ldots . + \frac{\partial x_1}{\partial q_n}.\delta q_n = \sum_{\zeta=1}^{n} \frac{\partial x_1}{\partial q_\zeta}.\delta q_\zeta$$

$$\delta x_2 = \frac{\partial x_2}{\partial q_1}.\delta q_1 + \frac{\partial x_2}{\partial q_2}.\delta q_2 + \ \ldots . + \frac{\partial x_2}{\partial q_n}.\delta q_n = \sum_{\zeta=1}^{n} \frac{\partial x_2}{\partial q_\zeta}.\delta q_\zeta$$

.

.

.

$$\delta x_{3N} = \frac{\partial x_{3N}}{\partial q_1}.\delta q_1 + \frac{\partial x_{3N}}{\partial q_2}.\delta q_2 + \ \ldots . + \frac{\partial x_{3N}}{\partial q_n}.\delta q_n = \sum_{\zeta=1}^{n} \frac{\partial x_{3N}}{\partial q_\zeta}.\delta q_\zeta$$

The above expressions may be written in a compact form as follows :

$$\delta x_i = \frac{\partial x_i}{\partial q_1}.\delta q_1 + \frac{\partial x_i}{\partial q_2}.\delta q_2 + \ \dots . + \frac{\partial x_i}{\partial q_n}.\delta q_n \ = \sum_{\zeta=1}^{n} \frac{\partial x_i}{\partial q_\zeta}.\delta q_\zeta \ (i = 1,2,3, \ \dots, 3N)$$

For a particle undergoing a displacement δr under the action of force F , the work done is given by

$$\delta W = F.\delta r = F_x \delta x + F_y \delta y + F_z \delta z$$

This work in the new notation can be expressed as

$$\delta W = \sum_{i=1}^{3} F_i \cdot \delta x_i$$

$and\ for\ a\ system\ consiisting\ of\ N\ particles$, we can write

$$\delta W = \sum_{i=1}^{3N} F_i \cdot \delta x_i$$

Substituting for δx_i we get

$$\delta W = \sum_{i=1}^{3N} F_i \sum_{\zeta=1}^{n} \frac{\partial x_i}{\partial q_\zeta} \delta q_\zeta$$

Hence,

$$\delta W = \sum_{\zeta} Q_\zeta \cdot \delta q_\zeta$$

where $Q_\zeta = \sum_i F_i \frac{\partial x_i}{\partial q_\zeta} = Generalised\ force$

D'Alembert's Principle.

Consider a system described by n generalised coordinates (q_1,q_2,q_3, … , q_n) undergoing a displacement such that it does not take any time and that it is consistent with the constraints of the system. Such displacements are called VIRTUAL because they do not represent actual displacements of the system. Since there is no actual motion of the system, the work done by the forces of constraints in such a virtual displacement is zero.

If a particle is constrained to move on the surface of a smooth sphere, then the force of constraint is equivalent to the reaction of the surface. In this case, the virtual displacement is taken at right angles to the direction of the force so that work done by the force during the virtual displacement is zero.

If δr_i is the virtual displacement of the i^{th} particle on which a resultant force F_i acts and the system is in equilibrium, then the virtual work done $F_i \cdot \delta r_i = zero$

The resultant force on the i^{th} particle is made up of two forces

$$F_i^a - \text{the applied force}$$

$$f_i - \text{the force of constraint}$$

Hence, we can write $F_i = F_i^a + f_i$

Therefore, we get $\sum \; F_i^a \cdot \delta r_i + \sum f_i \cdot \delta r_i = zero$

Let us assume that virtual work done by the forces of constraints is zero

$$\sum f_i \cdot \delta r_i = zero$$

The virtual displacements are such that the total work done by the forces of constraints is zero. The above equation will not hold good if the frictional forces are present. This is because, the frictional forces act in a direction opposite to that of the displacement.

Principle of Virtual Work states:

" ***Virtual Work done by the applied forces acting on a system in equilibrium is zero, provided no frictional forces are present***"

$$\sum_i F_i^a \cdot \delta r_i = zero$$

Most of the mechanical systems are not in static equilibrium but are in dynamic equilibrium. Hence, the principle must be modified to include dynamic systems also.

We can write $F_i = \dot{p}_i$ and hence $F_i \;\; - \dot{p}_i \;\; = Zero$. The system appears to be in dynamic equilibrium under the action of the applied force F_i and an equal and opposite 'effective force' $\dot{p}_i$. This effective force is called 'Kinetic Reaction'.

We can now generalise the principle of vitual work and can express:

$$\sum_i (\boldsymbol{F_i} \; - \boldsymbol{\dot{p}_i}) \cdot \boldsymbol{\delta r_i} = \boldsymbol{zero}$$

The above expression is called ***D'Alembert's Principle.***

In terms of geenralised coordinates, $\sum_j \boldsymbol{Q_j} \cdot \boldsymbol{\delta q_j} = \boldsymbol{zero}$

POTENTIAL and KINETIC ENERGIES.

The generalised coordinates q's need not have the dimensions of length and the generalised force $Q\zeta$ need not have the dimensions of force. But the product $Q_\zeta \delta q_\zeta$ must necessarily have the dimensions of work. The rectangular components of the force acting on a particle in a conservative force field is given by $F_i = -\left(\frac{\partial V}{\partial x_i}\right)$ where V is the potential energy function. The expression for generalised force in a conservative force field can be expressed as

$$Q_\zeta = \sum_i F_i \frac{\partial x_i}{\partial q_\zeta} = -\sum \frac{\partial V}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_\zeta} \quad \text{-----(1)}$$

Since V is a function of x's, we have

$$\frac{\partial V}{\partial q_\zeta} = \frac{\partial V}{\partial x_1} \cdot \frac{\partial x_1}{\partial q_\zeta} + \frac{\partial V}{\partial x_2} \cdot \frac{\partial x_2}{\partial q_\zeta} + \frac{\partial V}{\partial x_3} \cdot \frac{\partial x_3}{\partial q_\zeta} + \cdots + \frac{\partial V}{\partial x_{3N}} \cdot \frac{\partial x_{3N}}{\partial q_\zeta} = \sum \frac{\partial V}{\partial x_i} \cdot \frac{\partial x_i}{\partial q_\zeta} \quad \text{-----(2)}$$

Comparing equations (1) and (2), the generalised force is given in terms of potential by

$$Q_\zeta = -\frac{\partial V}{\partial q_\zeta}$$

The kinetic energy of the system of N particles is given by :

$$T = \sum_{i=1}^{3N} \frac{1}{2} m_i \dot{x}_i^2 \quad \text{------- (3)}$$

Generalised coordinates are expressed in terms of the coordinates x's and hence x's can be expressed in terms of q's as :

$$x_i = f_i(q_1, q_2, q_3, \dots, q_n)$$

Therefore,

$$\frac{dx_i}{dt} = \dot{x}_i = \frac{\partial x_i}{\partial q_1} . \dot{q}_1 + \frac{\partial x_i}{\partial q_2} . \dot{q}_2 + \ldots + \frac{\partial x_i}{\partial q_n} . \dot{q}_n = \sum_{\zeta=1}^{n} \frac{\partial x_i}{\partial q_\zeta} . \dot{q}_\zeta \quad \text{-------- (4)}$$

The quantities $\dot{q}$'s are called generalised velocities. Taking partial derivative of $\dot{x}$ wrt $\dot{q}_j$ we get $\frac{\partial \dot{x}_1}{\partial \dot{q}_\zeta} = \frac{\partial x_i}{\partial q_\zeta}$. This result may be remembered by looking at it as the cancellation of dots.

Multiplying by $\dot{x}_i$ *, we get* $\dot{x_i} \cdot \frac{\partial \dot{x_i}}{\partial \dot{q_\zeta}} = \dot{x_i} \cdot \frac{\partial x_i}{\partial q_\zeta}$.

Differentiating wrt 't' : $\frac{d}{dt}\left(\dot{x_i} \cdot \frac{\partial \dot{x}_i}{\partial \dot{q_\zeta}}\right) = \frac{d}{dt}\left(\dot{x_i} \cdot \frac{\partial x_i}{\partial q_\zeta}\right)$

$$= \ddot{x_i} \cdot \frac{\partial x_i}{\partial q_\zeta} + \dot{x_i} \cdot \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\zeta}\right)$$

$$= \ddot{x_i} \cdot \frac{\partial x_i}{\partial q_\zeta} + \dot{x_i} \cdot \frac{\partial}{\partial q_\zeta}\left(\frac{d x_i}{dt}\right)$$

$$= \ddot{x_i} \cdot \frac{\partial x_i}{\partial q_\zeta} + \dot{x_i} \cdot \frac{\partial \dot{x_i}}{\partial q_\zeta} \quad \text{----------} \; (5)$$

Also, $LHS = \frac{d}{dt}\left(\dot{x_i} \cdot \frac{\partial \dot{x}_i}{\partial \dot{q_\zeta}}\right) = \frac{d}{dt}\left(\frac{1}{2}\frac{\partial \dot{x}_i^{\,2}}{\partial \dot{q_\zeta}}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_\zeta}}\left(\frac{\dot{x_i}^2}{2}\right)\right)$

Equation (5) becomes $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_\zeta}}\left(\frac{\dot{x_i}^2}{2}\right)\right) = \ddot{x_i}\frac{\partial x_i}{\partial q_\zeta} + \frac{\partial}{\partial q_\zeta}\left(\frac{\dot{x_i}^2}{2}\right)$ ---------(6)

Differential Equation 4 wrt q_σ

$$\frac{\partial \dot{x_i}}{\partial q_\sigma} = \sum_{\zeta=1}^{n_1} \frac{\partial^2 xi}{\partial q_\sigma \partial q_\zeta} \cdot \dot{q_\zeta}$$

But $\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma}\right) = \sum_{\zeta=1}^{n_1} \frac{\partial^2 x_i}{\partial q_\sigma} \dot{q_\zeta}$

Comparing the above two equations :

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_\sigma}\right) = \frac{\partial \dot{x}_i}{\partial q_\sigma} = \frac{\partial}{\partial q_\zeta}\left(\frac{dx}{dt}\right) \quad \text{----------} \; (7)$$

This result shows that d and ∂ can be interchanged.

Lagrange's equations of motion.

Multiplying equation (6) above by m_i, the mass of i^{th} coordinate, we get :

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_\zeta}}\left(m_i \frac{\dot{x_i}^2}{2}\right)\right) = m_i \ddot{x_i} \frac{\partial x_i}{\partial q_\zeta} + \frac{\partial}{\partial q_\zeta}\left(m_i \frac{\dot{x_i}^2}{2}\right) \quad \text{-------} \; (8)$$

By Newton's law , $F_i = m_i \ddot{x_i}$

Substituting in the above equation and summing over all values of i, we get

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q_\zeta}}\left(\sum_{i=1}^{3N} m_i \frac{\dot{x_i}^2}{2}\right)\right) = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_\zeta} + \frac{\partial}{\partial q_\zeta}\left(\sum_{i=1}^{3N} m_i \frac{\dot{x_i}^2}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\zeta}\right) = Q_\zeta + \frac{\partial T}{\partial q_\zeta}$$

Hence,

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\zeta}\right) - \frac{\partial T}{\partial q_\zeta} = Q_\zeta}$$

These are called *Lagrange's Equations of Motion*.

$\frac{\partial T}{\partial \dot{q}_\zeta} = p_\alpha$ is called generalised momentum or conjugate momentum. Potential energy is a function of only q's.

Let L = T – V

Therefore, $\frac{\partial L}{\partial \dot{q}_\zeta} = \frac{\partial}{\partial \dot{q}_\zeta}(T - V) = \frac{\partial K}{\partial \dot{q}_\zeta}$

Hence, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\zeta}\right) - \frac{\partial T}{\partial q_\zeta} = Q_\zeta = -\frac{\partial V}{\partial q_\zeta}$

$$\Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\zeta}\right) - \frac{\partial}{\partial q_\zeta}(T - V) = 0$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\zeta}\right) - \frac{\partial L}{\partial q_\zeta} = 0}$$

This is called Lagrangian function of the system or simply Lagrangian.

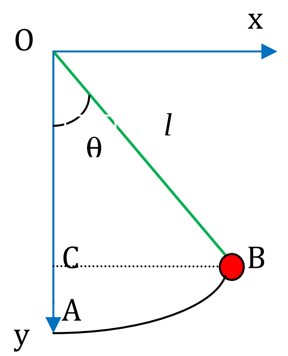
Lagrangian for Simple Pendulum.

Let θ be the angle made by the string with the vertical at an instant of time. The KE of the bob is given by

$T = \frac{1}{2}mv^2 = \frac{1}{2}ml\dot{\theta}^2$ where m = mass of the bob

PE of the bob about the mean position A : $V = mg(OA - OC) = mgl(1 - cos\theta)$

Lagrangian, L = T – V =$\frac{1}{2}ml\dot{\theta}^2 - mgl(1 - cos\theta)$



Lagrangian equation of motion :

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ as θ is the generalised coordinate.

$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ *and* $\frac{\partial L}{\partial \theta} = -mglsin\theta$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = ml^2\ddot{\theta}$$

Hence, the equation of motion is :

$$ml^2\ddot{\theta} + mglsin\theta = 0$$

$$\boldsymbol{\ddot{\theta} + \frac{g}{l}sin\theta = 0}$$

The Hamiltonian

Hamiltonian in terms of Lagrangian L is given by :

$$H = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L(q_\alpha, t) ------(1)$$

This must be expressed as a function of the generalised coordinates q_α and generalised momentum p_α. To accomplish this the generalised velocity $\dot{q_\alpha}$ must be eliminated from the above equation by using the Lagrange's equations. Then the function can be written as

$$H(p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n, t). -----(2)$$

This is the Hamiltonian of the system.

Hamilton's Equation

a. **When H does not contain t explicitly.**

Taking the differential of equation (1) :

$$dH = \sum p_\alpha d\dot{q}_\alpha + \sum \dot{q}_\alpha dp_\alpha - \frac{\partial L}{\partial q_\alpha}\mathrm{d}q_\alpha - \sum \frac{\partial L}{\partial \dot{q_\alpha}} d\dot{q_\alpha} ----(3)$$

Using the fact that $P_\alpha = \frac{\partial L}{\partial \dot{q_\alpha}}$ *and* $\dot{p_\alpha} = \frac{\partial L}{\partial q_\alpha}$ *, we get*

$$dH = \sum p_\alpha d\dot{q}_\alpha + \sum \dot{q}_\alpha dp_\alpha - \sum \dot{p_\alpha}\mathrm{d}q_\alpha - \sum p_\alpha \, d\dot{q_\alpha} = \sum \dot{q_\alpha} dp_\alpha - \sum \dot{p_\alpha} dq_\alpha --- (4)$$

Since H is expressed as a function of $q_\alpha$and p_α ,

$$dH = \sum \frac{\partial H}{\partial p_\alpha} dp_\alpha + \sum \frac{\partial H}{\partial q_\alpha} dq_\alpha ----(5)$$

Comparing equations (4) and (5)

$$\boldsymbol{\dot{q_\alpha} = \frac{\partial H}{\partial p_\alpha} \quad and \quad \dot{p_\alpha} = -\frac{\partial H}{\partial q_\alpha}} ----- \textbf{(6)}$$

b. When H contains t explicitly.

Equations (3), (4) and (5) will be modified as follows :

$dH = \sum \dot{q}_\alpha dp_\alpha + \sum p_\alpha d\dot{q}_\alpha - \sum \frac{\partial L}{\partial q_\alpha} dq_\alpha - \sum \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha - \frac{\partial L}{\partial t} dt - - - - (7)$

$$dH = \sum \dot{q}_\alpha dp_\alpha - \sum \dot{p}_\alpha dq_\alpha - \frac{\partial L}{\partial t} dt - - - - - - (8)$$

$$dH = \sum \frac{\partial H}{\partial p_\alpha} dp_\alpha + \frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial t} dt - - - - - - (9)$$

On comparing equations (8) and (9) :

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}; \ \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}; \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} - - - - (\mathbf{10})$$

Thus, in terms of the Hamiltonian, the equations of motion of the system can be written in the symmetrical form as shown in equations (6) or (10). These are called **Hamilton's Equations of Motion.**

Hamiltonian for Conservative Systems.

When H is independent of t explicitly, then it can be shown that it is a constant and is equal to the total energy of the system. We have :

$$dH = \sum \dot{q}_\alpha dp_\alpha - \sum \dot{p}_\alpha dq_\alpha$$

$$Therefore, \frac{\partial H}{\partial t} = \sum \dot{q}_\alpha \dot{p}_\alpha - \sum \dot{p}_\alpha \dot{q}_\alpha = 0$$

$$Hence, H = constant, say\ E$$

From Euler's theorem on Homogeneous function

$$\sum \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T where\ T\ is\ Kinetic\ energy$$

Since $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial K}{\partial \dot{q}_\alpha}$ (assuming that potential energy does not depend upon $\dot{q}_\alpha$)

We have $\sum p_\alpha \dot{q}_\alpha = \sum \frac{\partial K}{\partial \dot{q}_\alpha} d\dot{q}_\alpha = 2K$

Hence,

$H = \sum p_\alpha \dot{q}_\alpha - L = 2T - T + U = T + U = E, the\ total\ energy\ of\ the\ system$

If the system is conservative, the Hamiltonian H can be expressed as the total energy (Kinetic +Potential) of the System.

$$\boldsymbol{H = T + U}$$