SOME ANALYTICAL AND NUMERICAL SOLUTIONS OF BOUNDARY LAYER EQUATIONS FOR SIMPLE FLOWS

Report of the work done under the UGC Minor Research Project in Mathematics

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Declaration

I hereby declare that the matter embodied in this report is the result of the investigations carried out by me in the Department of Mathematics, Vijaya College, R V Road, Basavanagudi, Bangalore 560 004, under the able guidance of **Dr Achala L Nargund**. Head and Coordinator, Post Graduate Department of Mathematics and Research Center in Applied Mathematics, MES College, Malleswaram, Bangalore 560 003.

No part of the subject matter presented in this report has been submitted for the award of any Degree, Diploma, associateship, fellowship, etc of any University or Institute.

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Bangalore, India S B Sathyanarayana

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To my Beloved Parents

whose love and forbearance made it possible

for me to saver both the tears and joy of

my education to the full................

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Chapter 1

Introduction

1.1 Principles of Fluid Mechanics

The liquids and gases are the states of a matter that come under the same category as ”fluids”. A Fluid is a substance that deforms continuously when subjected to stresses however small it may be. Such continuous deformation constitutes a flow. The theory of fluid flow (incompressible or compressible fluid) is based on the Newtonian mechanics. The concept of continuum is a kind of idealization of continuous description of matter where the properties of the matter such as density, viscosity, thermal conductivity temperature etc are considered as continuous functions of space variables and time.

All mathematical models of the real world problems occurring in fluid mechanics obey the following fundamental conservation laws

1. Conservation of Mass

2. Conservation of Momentum

3. Conservation of Energy

It is useful to classify the type of fluid flow into the following

Uniform flow:

The fluid flow is said to be uniform If the flow velocity is the same both in magnitude and direction at every point in the fluid.

Non-uniform flow:

The fluid flow is said to be non uniform if the velocity is not the same at every point at a given instant.

Steady flow:

A steady flow is a fluid flow in which the conditions (velocity, pressure and cross-section) may differ from point to point but do not change with time.

Unsteady flow:

A unsteady flow is a fluid flow in which the conditions change with time at any point in the fluid.

Incompressible flow

Incompressible flow refers to a flow in which the density is constant within a fluid parcel - an infinitesimal volume that moves with the velocity of the fluid. Incompressible flow implies that the density remains constant within a parcel of fluid which moves with the fluid velocity.

Compressible flow

Compressible flow is the area of fluid mechanics that deals with fluids in which the fluid density varies significantly in response to a change in pressure. A change in density brings an additional variable into the analysis and this introduces another variable (temperature), and so a fourth equation (such as the ideal gas equation) is required to relate the temperature to the other thermodynamic properties in order to fully describe the flow.

Thus in case of compressible flow fluid models can be solved by considering not only the equations from conservation of mass and momentum but also the principle of conservation of energy. The most distinct differences between the compressible and incompressible flow models are that the compressible flow model allows for the existence of shock waves and choked flow.

Compressible and incompressible fluid flow problems modeling involves nonlinear differential equations. Due to non availability of standard methods to find analytical solutions, many problems are solved using approximate as well as numerical methods.

1.2 Boundary Layer Theory

The boundary layer theory [1] which was first developed by L. Prandtl in 1904 gave a convincing explanation for motion of fluid around objects and this led to major advances in fluid dynamics. The detailed analysis of the flow within the boundary layer region is very important for many engineering problems and aerodynamics. If a fluid flows in the presence of an obstacle, then the obstacle will experience two types of forces,

1. drag force in the direction of motion of the fluid,

2. lift force in a direction normal to the flow direction.

These two forces are produced by tangential and normal stresses. The shearing stress i.e., the drag due to tangential stress is called friction or skin friction or viscous drag. The drag due to normal stress is called pressure drag.Thus flows constrained by solid surfaces can typically be divided into two regions as below,

1. **Boundary Layer Region**

   Flows are near a bounding surface with significant velocity gradients normal to the solid body and shear stresses in this region are predominant.

2. **Potential Flow Region**

   Flows far from bounding surface with negligible velocity gradients, negligible shear stresses where inertia effects are important.

Description of Boundary Layer

Boundary layers are the thin fluid layers adjacent to the surface of a body in which strong viscous effects exist. Consider the nature of flow field that would exist around an arbitrary body at a Reynolds number that is not small or of order unity. The nature of such a flow field is known from information gathered from large number of experiments.The streamline originates at the front stagnation point and moves downstream near the top and bottom surfaces of the body. In the potential flow region, relative to the body, the velocity gradients are not large, and so viscous effects are negligible. Then, if compressible effects may be ignored, the fluid may be considered ideal.

Boundary layers may be either laminar, or turbulent depending on the value of the Reynolds number. For lower Reynolds numbers, the boundary layer is laminar and the stream wise velocity changes uniformly as one move away from the wall. For higher Reynolds numbers, the boundary layer is turbulent and the stream wise velocity is characterized by unsteady swirling flows inside the boundary layer.

The external flow reacts to the edge of the boundary layer just as it would to the physical surface of an object. So the boundary layer gives any object an ”effective” shape which is usually slightly different from the physical shape. Since the flow in the boundary has very low energy and is more easily driven by changes in pressure, the boundary layer may lift off or ”separate” from the body and create an effective shape much different from the physical shape.

Boundary Layer Thickness

The thickness of the velocity boundary layer is normally defined as the distance from the solid body at which the viscous flow velocity is 99% of the free stream velocity. The boundary layer (velocity layer) thickness, δ, is the distance across a boundary layer from the wall to a point where the flow velocity u has essentially reached the ’free stream’ velocity, U. The distance is measured along normal to the wall, and the point where the flow velocity is essentially that of the free stream.

Flow problems in Boundary layer theory

Laminar boundary layers are classified according to their structure and the circumstances under which they are created.

1. Stokes boundary layer is a layer in which the thin shear layer develops on an oscillating body

2. Blasius boundary layer refers to the well-known similarity solution near an attached flat plate held in an oncoming unidirectional flow.

Compressible Boundary layers

The boundary layer theory which was first developed for laminar incompressible fluids was later extended to compressible flows also. The physical ideas underlying the boundary - layer concept are translatable to compressible flows. In this case, the thermal effects during the flow and heat transfer in the boundary layer exist due to the variation of density and viscosity with temperature. Thus both the physics and mathematics concepts become complicated. Consequently, flow problems of laminar compressible boundary layers have not been investigated to the same extent as those for incompressible flows.

1.3 Mathematical Methods

Non-linear Problems and Perturbation

Many of the problems in modeling that arise in technological and industrial situations are highly non-linear. As a result, it is often difficult to obtain analytical solutions to these problems. During the last century, perturbation methods have often been used to obtain solutions to these problems. These methods, however, are typically dependent on the presence of a small or large parameter; consequently, perturbation methods often have some restriction to provide accurate results for moderate to large (or small) values of the parameters.Asymptotic analysis provide solutions to problems for all parameters of interest. Liao [7] gives an example that effectively illustrates the limitations of traditional perturbation methods: the problem of a body of mass m falling freely through space with a velocity that varies with time under the influence of gravity g and air resistance. He then goes on to give an alternative technique known as the homotopy analysis method (HAM), which, for this particular problem, does not suffer from the said limitation.

Similarity solution

In this method, the number of independent variables is reduced by using appropriate combination of original independent variables as new independent variables called similarity variables. This method is called the similarity method which is used for obtaining the exact solutions of Partial Differential Equations.

Using the concept of dimensional analysis and scaling laws, by looking at the physical effects present in a system, we find that the solution of the system is not fixed on a natural length scale (time scale), but depends on space (time).

It is then necessary to construct a length scale (time scale) using time (space) and the other dimensional quantities present such as viscosity These constructs are not 'guessed' but are derived immediately from the scaling of the governing equations.

Numerical methods are generally used to solve systems of nonlinear ordinary differential equations which are obtained as a reduction of most boundary-layer models. It is however interesting to find approximate analytical solutions to boundary layer problems.

Analytical methods have significant advantages over numerical methods in providing relation between derived quantities. These solutions are analytic, verifiable and rapidly convergent approximation which have wider range of applications.

An analytical solution is exact and it is a formula that can be used for any situation and can be used to account for the errors in measurements. The formula will show the intrinsic relationship between the variables and therefore can be manipulated into different forms.There are large number of approximate solution procedures for the solution of both linear and nonlinear equations. These include, integral transforms, perturbation methods, Series analysis etc. A numerical solution is almost always approximate and is unique to the given situation. In general it is a rough ”estimate” of the right answer based on mathematical techniques and may therefore have a higher degree of uncertainty. For linear and some other system, Lax equivalence theorem (for finite difference scheme) shows some numerical schemes to be exact.

1.4 Homotopy Analysis Method

In 1992 Shijun Liao, in his Ph.D thesis, suggested a powerful method called Homotopy analysis method to solve almost all nonlinear ordinary differential equations analytically. The basic concept behind this method is the concept of homotopy from topology. This method generates a convergent series by utilizing homotopy-Maclaurin series which is almost analogous to Adomian polynomials.

The Homotopy analysis method (HAM) based on series approximation was first developed by Liao [2,3] for strongly nonlinear problems. HAM uses base functions to obtain series solutions to boundary-layer equations. Liao [4] has given comparison between HAM and Homotopy perturbation method and showed that HAM is general method, Homotopy perturbation method is only a special case of the HAM. Liao et al. [5] have studied the temperature distributions for a laminar viscous flow over a semi-infinite plate by HAM. Liao [6] gave the relationship between the HAM and Euler transform and showed that Euler transform is equivalent to HAM but HAM is more powerful then Euler transform. HAM has proven to be very efficient in solving the nonlinear boundary value problems with infinite domain.

The Homotopy analysis method (HAM) that has been advanced by Liao [2,3,4,5,6,7]. The basic idea of the Homotopy analysis method (HAM) is to produce a succession of approximate solutions that tend to the exact solution of the problem. The presence of auxiliary parameters and functions in the approximate solution results in the production of a family of approximate solutions rather than the single solution produced by traditional perturbation methods. By varying these auxiliary parameters and functions, it is possible to adjust the region and rate of convergence of the series solution .

1.5 Need for the study

Boundary Layer Theory has a lot of practical applications such as extrusion of plastic sheets, rolling and manufacturing of plastic films, cooling of metallic plates and boundary layer flow over heat treated materials between feed roll, a windup roll and in Aerodynamics. All Boundary value problems are represented by non linear partial differential equations. These problems can be reduced to non linear ordinary differential equations.

Homotopy analysis method is a method used to find analytical solution which needs computers. High end computers are therefore helpful to find analytical solutions of high accuracy.

1.6 Scope of the Research Work presented in this Project

Almost all real world problems are non-linear and many can't be solved with analytic techniques. In these cases, numerical methods (like finite element analysis,finite difference scheme, Runge kutta merson methods, etc) are the only choice and can be solved effectively by means of sophisticated computers with good accuracy. In spite of that, the searches for exact analytic solutions of nonlinear partial differential equations continue unceasingly. This is because it is difficult to get general idea of the problem purely from numbers; in any case they must be compared with some known exact solutions of specific cases . Any phenomenon can be completely understood by analysis of physical system which enriches the mathematics involved in it.

The main objective is to get analytic solutions for nonlinear differential equations arising in fluid dynamics, in particular boundary layer region and validate the method used to obtain analytic solution by comparing it to their numerical solutions.

Chapter 2

Homotopy Analysis Method

2.1 Details of Homotopy Analysis Method

Consider a nonlinear differential equation

$$N[u(t)] = 0 \tag{2.1.1}$$

where N is a nonlinear operator, t denotes the time, and u(t) is an unknown variable. Let $u_0(t)$ denote an initial approximation of u(t) and L denote an auxiliary linear operator.

We construct the Homotopy as

$$H[\phi(t;q);q] = (1-q)L[\phi(t;q) - u_0(t)] + qN[\phi(t;q)], \tag{2.1.2}$$

Where $q \, \epsilon \, [0,1]$ is an embedding parameter and $\phi(t;q)$ is a function of t and q. When q = 0 and q = 1, equation (2.1.2) becomes

$$H[\phi(t;q);q]|_{q=0} = L[\phi(t;0) - u_0(t)] \tag{2.1.3}$$

$$H[\phi(t;q);q]|_{q=1} = N[\phi(t;1)] \tag{2.1.4}$$

respectively.

It is clear that $\phi(t;0) = u_0(t)$is the solution of the equation

$$H[\phi(t;q);q]|_{q=0} = 0$$

and $\phi(t;1) = u(t)$ is therefore obviously the solution of the equation

$$H[\phi(t;q);q]|_{q=1} = 0$$

As the embedding parameter q increases from 0 to 1, the solution $\phi(t;q)$ varies from the initial approximation $u_0(t)$ to the solution u(t) of equation N[u(t)] = 0.

In topology, such a kind of continuous variation or deformation is called as homotopy which is used in Homotopy Analysis Method .

By using Maclaurin series for $\phi(t;q)$, it can be expressed as

$$\phi(t,q) = \phi(t,0) + \sum_{k=1}^{+\infty} \frac{q^k}{k!} \left. \frac{\partial^k \phi(t,q)}{\partial q^k} \right|_{q\,=\,0}. \tag{2.1.5}$$

We take $\phi(t,0) = \phi_0(t) = f_0(t)$ and Define

$$f_k(t) = \frac{1}{k!} \left. \frac{\partial^k \phi(t,q)}{\partial q^k} \right|_{q\,=\,0} (k > 0) \tag{2.1.6}$$

Equations (2.1.5) and (2.1.6) become

$$\phi(t,q) = f_0(t) + \sum_{k=1}^{+\infty} f_k(t) q^k. \tag{2.1.7}$$

Now the solution is a series $f_k(t)$. In order to calculate $f_k(t)$, we set homotopy equal to zero in equation (2.1.2) and differentiating k times about the embedding parameter q and applying Leibnitz theorem, setting $q = 0$ and dividing by $k!$, we get

$$L[\phi_k - \chi_k \phi_{k-1}] = hR_k(\eta), \tag{2.1.8}$$

where $\chi_k = \begin{cases} 0 & when\, k \leq 1 \\ 1 & when\;\; k > 1 \end{cases}$

$$R_k(t) = h\, H(t) \frac{1}{(k-1)!} \, \frac{\partial^{k-1} N[\phi(t;q)]}{\partial q^{k-1}} \bigg|_{q=0}. \tag{2.1.9}$$

Obviously, the convergence region of the above series depends upon the auxiliary linear operators L, the initial guess $u_0(t)$ and the non-zero auxiliary parameter h. If all of them are selected so that (2.1.7) converge at $p = 1$, and thus (2.1.7) can be written as

$$f(t) = f_0(t) + \sum_{k=1}^{+\infty} f_k(t), \tag{2.1.10}$$

where f_k are unknowns to be obtained and this solution will be valid whenever the series is convergent.

It is observed that the convergence region by HAM is larger than the other methods. The region of convergence and the rate of convergence can be adjusted by varying L, $u_0(t)$ and h.

2.2 Guidelines for Developing Solutions

The Homotopy Analysis Method gives us a great deal of freedom for the selection of an appropriate linear operator, auxiliary function, and initial approximations to develop a convergent solution to given nonlinear differential equation.

A set of base functions $\{ e_n(t) | \, n = 0, 1, 2,...\}$ are chosen first such that they satisfy the initial or boundary conditions of a problem and by considering the physical interpretation and expected asymptotic behaviour of the solution.

By the **rule of solution expression**, we can write the solution as

$$f(t) = \sum_{0}^{\infty} c_n e_n(t),$$

where c_n are constants. This rule will also guide us to choose linear operator L and auxiliary function $H(t)$.

For the sake of completeness of the solution **the rule of coefficient ergodicity** also should be applied.

The third rule, **the rule of solution existence** requires the selection of the linear operator, auxiliary function and initial approximation gives analytical solution to each one of the deformation equations.

2.3 Illustration of Homotopy Analysis Method

Introduction

When a sheet of polymer is extruded continuously from a die, it entrains the ambient fluid and a boundary layer develops. Such a boundary layer is markedly different from that in the Blasius flow past a flat plate in that the boundary layer grows in the direction of the motion of the sheet, starting at the die.

2.4 Governing equations

Consider the steady two-dimensional incompressible flow of an electrically conducting viscous fluid past a nonlinearly semi-infinite stretching sheet. The governing boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = v_m \frac{\partial^2 u}{\partial y^2} \tag{2.4.2}$$

where x and y are distances along and perpendicular to the sheet, respectively, u and v are components of the velocity along x and y directions respectively, v_m is kinematic viscosity. The corresponding boundary conditions,

$$u(x,0) = \text{ax} + cx^2, v(x,0) = 0 \tag{2.4.3}$$

$$u \to 0 \text{ as } \quad y \to \infty \tag{2.4.4}$$

where a and c are constants .

We introduce the similarity transformations

$$\eta = \sqrt{\frac{a}{\nu_m}} y, \quad u = ax\, f'(\eta) + cx^2 g'(\eta), \tag{2.4.5}$$

$$v = -\sqrt{a\nu_m} f(\eta) \text{-} \frac{2cx}{\sqrt{a\,/\nu_m}} g(\eta) \tag{2.4.6}$$

to equations (2.4.1)and (2.4.2) as follows

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{a}{\nu_m}} \tag{2.4.7}$$

$$\frac{\partial u}{\partial x} = a\, f'(\eta) + 2cxg'(\eta),$$

$$\frac{\partial v}{\partial y} = -\sqrt{a\nu_m} f'(\eta) \frac{\partial \eta}{\partial y} \text{-} \frac{2cx}{\sqrt{a\,/\nu_m}} g'(\eta) \frac{\partial \eta}{\partial y},$$

$$\frac{\partial v}{\partial y} = -\sqrt{a\nu_m} f'(\eta) \sqrt{\frac{a}{\nu_m}} \text{-} \frac{2cx}{\sqrt{a\,/\nu_m}} g'(\eta) \sqrt{\frac{a}{\nu_m}} \ ,$$

$$\frac{\partial v}{\partial y} = -af'(\eta) \text{-} 2cxg'(\eta) \ ,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial y} = ax\, f''(\eta) \frac{\partial \eta}{\partial y} + cx^2 g''(\eta) \frac{\partial \eta}{\partial y},$$

$$\frac{\partial u}{\partial y} = ax\, f''(\eta) \sqrt{\frac{a}{\nu_m}} + cx^2 g''(\eta) \sqrt{\frac{a}{\nu_m}} \ ,$$

$$\frac{\partial^2 u}{\partial y^2} = ax\, f'''(\eta) \left(\sqrt{\frac{a}{\nu_m}} \right)^2 + cx^2 g'''(\eta) \left(\sqrt{\frac{a}{\nu_m}} \right)^2 \ ,$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a^2 x}{\nu_m} f'''(\eta) + \frac{acx^2}{\nu_m} g'''(\eta),$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v_m \frac{\partial^2 u}{\partial y^2},$$

$$\left[ax\, f'(\eta) + cx^2 g'(\eta)\right] \left[a\, f'(\eta) + 2cxg'(\eta)\right] +$$

$$\left[-\sqrt{a\nu_m} f(\eta) \text{-} \frac{2cx}{\sqrt{a\,/\nu_m}} g(\eta) \right] \left[ax\, f''(\eta) \sqrt{\frac{a}{\nu_m}} + cx^2 g''(\eta) \sqrt{\frac{a}{\nu_m}} \ \right],$$

$$= \nu_m \left[\frac{a^2 x}{\nu_m} f'''(\eta) + \frac{acx^2}{\nu_m} g'''(\eta) \right]$$

$$a^2 x \left((f')^2 - ff'' - f''' \right)$$

$$+ acx^2 \left(3f'g' - 2f''g - fg'' - g''' \right) + 2c^2 x^3 \left((g')^2 - gg'' \right) = 0, \tag{2.4.8}$$

This reduces Equations. (2.4.1) - (2.4.4) to a system of dimensionless nonlinear ordinary differential equations

$$f''' + ff'' - (f')^2 = 0, \tag{2.4.9}$$

$$g''' + fg'' + 2f''g - 3f'g' = 0, \tag{2.4.10}$$

subject to boundary conditions,

$$f(0) = 0, \; f'(0) = 1, \; f'(+\infty) = 0, \tag{2.4.11}$$

$$g(0) = 0, \; g'(0) = 1, \; g'(+\infty) = 0. \tag{2.4.12}$$

where f and g are functions related to the velocity field, N is magnetic parameter. The primes denote differentiation with respect to η.

2.5 Method of Solution

The solution of equation (2.4.5) under (2.4.7) is considered for explaining homotopy analysis method.

We first select the auxiliary linear operator

$$L = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2} \tag{2.5.1}$$

Then, we construct a family of partial differential equations

$$(1\text{-}p)L[F(\eta,p)\text{-}f_0(\eta))]=hp\left\{\frac{\partial^3 F}{\partial \eta^3}+F\frac{\partial^2 F}{\partial \eta^2}-\left(\frac{\partial F}{\partial \eta}\right)^2\right\} \tag{2.5.2}$$

with boundary conditions

$$F(0,p)=0, F_\eta(0,p)=1, F_\eta(+\infty,p)=0 \tag{2.5.3}$$

where F_η denotes the first-order derivative of $F(\eta, p)$ with respect to η,

$p \in$ [0,1] is the embedding parameter,

$h \neq 0$ is an non zero auxiliary parameter,

$f_0(\eta)$ is the initial guess approximations of$f(\eta)$,

We choose $f_0(\eta)$ as follows in accordance with boundary conditions (2.5.3)

$$f_0(\eta)=1-e^{-\eta} \tag{2.5.4}$$

When p = 0, we have the solution

$$F(\eta,0)=f_0(\eta), \tag{2.5.5}$$

and

When p = 1, equations (2.5.2) - (2.5.3) is the same as (2.4.5) - (2.4.7) so that

$$F(\eta,1)=f(\eta), \tag{2.5.6}$$

Thus as p increases from 0 to 1, the solution varies from the initial guess $f_0(\eta)$ to the solution $f(\eta)$.

The initial guess approximation $f_0(\eta)$, the auxiliary linear operator L and the auxiliary parameters h are assumed to be selected such that equations (2.5.2) - (2.5.3) have solutions at each point $p \in$ [0,1] and also $F(\eta, p)$ can be expressed in Maclaurin series

$$F(\eta, p) = F(\eta, 0) + \sum_{k=1}^{+\infty} \frac{p^k}{k!} \left. \frac{\partial^k F(\eta, p)}{\partial p^k} \right|_{p=0} \tag{2.5.7}$$

Defining

$$F(\eta, 0) = f_0(\eta) = \phi_0(\eta), \tag{2.5.8}$$

$$\phi_k(\eta) = \frac{1}{k!} \left. \frac{\partial^k F(\eta, p)}{\partial p^k} \right|_{p=0} (k > 0), \tag{2.5.9}$$

We have due to (2.5.7) - (2.5.9) that

$$F(\eta, p) = \phi_0(\eta) + \sum_{k=1}^{+\infty} \phi_k(\eta) p^k, \tag{2.5.10}$$

Obviously, the convergence region of the above series depends upon the auxiliary linear operator L and the non-zero auxiliary parameter h . If all of them are selected so that (2.5.10) converge at p = 1, Substituting (2.5.6) in (2.5.10), We obtain

$$f(\eta) = \phi_0(\eta) + \sum_{m=1}^{+\infty} \phi_m(\eta), \tag{2.5.11}$$

Here $\phi_m(\eta)$ are unknowns to be solved

Differentiating m times the two sides of equations (2.5.2) that is

$$(1-p)L[F(\eta,p)-f_0(\eta))] = hp \left\{ \frac{\partial^3 F}{\partial \eta^3} + F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2 \right\}, \tag{2.5.12}$$

about the embedding parameter p ,we have

$$\frac{d^m}{dp^m}\left((1-p)L[F(\eta,p)-f_0(\eta))]\right) = h\frac{d^m}{dp^m}\left(p\left\{ \frac{\partial^3 F}{\partial \eta^3} + F\frac{\partial^2 F}{\partial \eta^2} - \left(\frac{\partial F}{\partial \eta}\right)^2 \right\}\right),$$

By Using Leibnitz theorem,

$$\frac{d^m}{dx^m}(\mathrm{uv})=\mathrm{u}_m v +^m C_1 u_{\text{m-1}} v_1 +^m C_2 u_{\text{m-2}} v_2 + \cdots +^m C_r u_{\text{m-r}} v_r + \cdots + \mathrm{uv}_m,$$

$$\frac{d^m}{dx^m}(\mathrm{uv})=\sum_{k=0}^{m}(^m C_k u_{\text{m-k}} v_k),$$

Setting p = 0, dividing by m! and using m! = m (m-1)!

$$\mathrm{L}[\phi_m\text{-}\chi_m\phi_{\mathrm{m}-1}]=\mathrm{hR}_m(\eta), \tag{2.5.13}$$

Where

$$\chi_m = \begin{cases} 0 & when\, m \leq 1 \\ 1 & when\, m > 1 \end{cases} ,$$

$$R_m(\eta) = \phi'''_{m-1}(\eta) + \sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\phi''_k(\eta) - \sum_{k=0}^{m-1} \phi'_{m-1-k}(\eta)\phi'_k(\eta), \tag{2.5.14}$$

with boundary conditions

$$\phi_m(0) = \phi'_m(0) = \phi'_m(+\infty)=0, \tag{2.5.15}$$

We obtain linear equations forϕ_m (m $\geq$ 1),

In (2.5.13), putting m = 1, h = 1

$$\chi_m = 0$$

$$\left(\frac{\partial^3}{\partial\eta^3} + \frac{\partial^2}{\partial\eta^2}\right)\phi_1=\phi'''_0(\eta) + \phi_0(\eta)\phi''_0(\eta) - \phi'_0(\eta)\phi'_0(\eta),$$

$$\phi_1=0, \tag{2.5.16}$$

In (2.5.13), putting m = 2 , h = 1

$$\chi_m = 1,$$

$$\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_2-\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_1{=}\phi_1{'''}(\eta)+\phi_1(\eta)\phi_1{''}(\eta)-\phi_1'(\eta)\phi_1'(\eta),$$

$$\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_2{=}2\phi_1{'''}(\eta)+\phi_1{''}(\eta)+\phi_1(\eta)\phi_1{''}(\eta)-\phi_1'(\eta)\phi_1'(\eta)-N\phi_1'(\eta)$$

$$\phi_2{=}0 \tag{2.5.17}$$

In (2.5.13), putting m = 3 , h = 1

$$\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_3-\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_2{=}\phi_2{'''}(\eta)+\phi_2{''}(\eta)+\phi_2(\eta)\phi_0{''}(\eta)+$$

$$\phi_1(\eta)\phi_1{''}(\eta)+\phi_0(\eta)\phi_2{''}(\eta)-\phi_2'(\eta)\phi_0'(\eta)-\phi_1'(\eta)\phi_1'(\eta)-\phi_0'(\eta)\phi_2'(\eta)$$

$$\left(\frac{\partial^3}{\partial\eta^3}+\frac{\partial^2}{\partial\eta^2}\right)\phi_3{=}2\phi_2{'''}(\eta)+\phi_2{''}(\eta)+\phi_2(\eta)\phi_0{''}(\eta)+\phi_1(\eta)\phi_1{''}(\eta)+$$

$$\phi_0(\eta)\phi_2{''}(\eta)-2\phi_2'(\eta)\phi_0'(\eta)-\phi_1'(\eta)\phi_1'(\eta)$$

$$\phi_3{=}0 \tag{2.5.18}$$

We have

$$f(\eta)=\phi_0+\phi_1+\phi_2+\phi_3+\cdots$$

$$f(\eta)=1\text{-}e^{-\eta} \tag{2.5.19}$$

It is observed the final solution f(η) obtained agrees with the initial approximation $f_0(\eta)$, which satisfy the differential equation and boundary conditions. Other initial approximation $f_0(\eta)$ can also be chosen to suit our requirements.

We can generate large number of terms solving the linear equations by MATHEMATICA, Because of availability of large number of coefficients, we can use Pade's approximation to test the convergence of this series and is found convergent.

Chapter 3

Analytical Solution of Flow Past A Permeable Shrinking Sheet

3.1 Governing Equations

Consider the Newtonian fluid past a permeable shrinking sheet which is electrically conducting and magnetic field in applied perpendicular to the fluid flow. A boundary layer is formed due to the flow . Heat transfer is due to internal heat absorption or generation. The sheet coincides with x - axis and flow is confined to region $y > 0$. The governing boundary layer equations [8] are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_m \frac{\partial^2 u}{\partial y^2} - \frac{\sigma {B_o}^2}{\rho} u, \tag{3.1.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho C_p}(T - T_\infty) \tag{3.1.3}$$

where x and y are distances along and perpendicular to the sheet, u and v are components of the velocity along x and y directions, respectively, $\nu_m = \frac{\mu}{\rho}$ is kinematic viscosity,

ρ is fluid density, σ is electrical conductivity, B_o is the strength of the magnetic field. T is the temperature, T_∞ is free stream temperature, κ is thermal conductivity of the fluid, Q_0 is volumetric rate of heat absorption or generation.

The corresponding boundary conditions are,

$$u(x,0) = U_w = -cx,\ v(x,0) = -v_w, \tag{3.1.4}$$

$$u \to 0\ as\ \ y \to \infty.$$

$$T(x,0) = T_w, \tag{3.1.5}$$

$$T \to T_\infty\ as\ \ y \to \infty.$$

where $c > 0$ is the shrinking sheet. T_w is temperature of the sheet, v_w represents the wall mass suction through the porus sheet.

The stream function ψ introduce as below

$$u = \frac{\partial \psi}{\partial y}\ \ and\ \ v = -\frac{\partial \psi}{\partial x}, \tag{3.1.6}$$

The dimensionless variables for ψ and T are introduced as

$$\psi = -\sqrt{c\nu}xf(\eta), \tag{3.1.7}$$

and

$$T = T_\infty + (T_w - T_\infty)\theta(\eta), \tag{3.1.8}$$

The similarity variable η is given by

$$\eta = y(\frac{c}{\nu})^{1/2}, \tag{3.1.9}$$

Using (3.1.6) to (3.1.9) in (3.1.1),(3.1.2) and (3.1.3), we obtain the following ordinary differential equations in self similar forms as

$$f''' + \, ff'' \, - \, (f')^2 - M^2 f' = 0, \tag{3.1.10}$$

$$\theta'' + Pr \, (f\theta' - \lambda\theta) = 0, \tag{3.1.11}$$

with boundary conditions

$$f(0) = s, f'(0) = -1, f'(+\infty) = 0, \tag{3.1.12}$$

$$\theta(0) = 0, \theta(+\infty) = 0. \tag{3.1.13}$$

where f and g are functions related to the velocity field, M is magnetic parameter. The primes denote differentiation with respect to η. $s = \frac{\nu_m}{(\mathrm{cv})^{1/2}} (> 0)$ is mass suction parameter.

3.2 Homotopy Analysis Method

To apply the homotopy analysis method [1,2] to the problem considered, we first select the auxiliary linear operator$L$ as

$$L = \frac{\partial^3}{\partial\eta^3} + \frac{\partial^2}{\partial\eta^2}. \tag{3.2.1}$$

Then, we construct a family of partial differential equations as follows

$$(1-p)L[F(\eta,p) - f_0(\eta)] = hp\left\{\frac{\partial^3 F}{\partial\eta^3} + \, F\frac{\partial^2 F}{\partial\eta^2} \, - \, \left(\frac{\partial F}{\partial\eta}\right)^2 - M^2\frac{\partial F}{\partial\eta}\right\}, \tag{3.2.2}$$

$$(1-p)L[G(\eta,p) - g_0(\eta)] = hp\left\{\frac{\partial^2 G}{\partial\eta^2} + Pr\,F\frac{\partial G}{\partial\eta} \, - Pr\lambda G\right\}, \tag{3.2.3}$$

with boundary conditions

$$F(0,p) = 0, F_\eta(0,p) = 1, F_\eta(\infty,p) = 0, \tag{3.2.4}$$

$$G(0,p) = 0, G(\infty,p) = 0, \tag{3.2.5}$$

where F_η denotes the first-order derivative of $F(\eta,p)$ with respect to η, G_η denotes the first-order derivative of $G(\eta,p)$ with respect to η, $p \,\epsilon\, [0,1]$ is the embedding parameter, $h \neq 0$ is an non zero auxiliary parameter.

We derive the initial guess $f_0(\eta)$ and $g_0(\eta)$ as follows in accordance with boundary conditions (3.2.4) and (3.2.5) as

$$f_0(\eta) = s - 1 + e^{-\eta}, \tag{3.2.6}$$

$$g_0(\eta) = e^{-\eta}. \tag{3.2.7}$$

When p = 0, we have the solution

$$F(\eta,0) = f_0(\eta), \tag{3.2.8}$$

$$G(\eta,0) = g_0(\eta). \tag{3.2.9}$$

When p = 1, equations (3.2.2) - (3.2.5) are the same as (3.1.10) - (3.1.13), respectively, so that

$$F(\eta,1) = f(\eta), \tag{3.2.10}$$

$$G(\eta,1) = g(\eta). \tag{3.2.11}$$

Thus as p increases from 0 to 1, the solution varies from the initial guess approximation $f_0(\eta)$ and $g_0(\eta)$ to the solution $f(\eta)$ and $g(\eta)$ respectively.

The initial guess approximation $f_0(\eta)$ and $g_0(\eta)$, the auxiliary linear operator L and the auxiliary parameter h are assumed to be selected such that the equations (3.2.2) and (3.2.3) with boundary condition (3.2.4) and (3.2.5) have solution at each point of $p \,\epsilon\, [0,1]$.

The solution can be expressed in Maclaurin series as

$$F(\eta,p)=F(\eta,0)+\sum_{k=1}^{+\infty}\frac{p^k}{k!}\left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p\,=\,0}, \tag{3.2.12}$$

$$G(\eta,p)=G(\eta,0)+\sum_{k=1}^{+\infty}\frac{p^k}{k!}\left.\frac{\partial^k G(\eta,p)}{\partial p^k}\right|_{p\,=\,0}. \tag{3.2.13}$$

Defining

$$\phi_0(\eta)=F(\eta,0)=f_0(\eta),$$

$$\phi_k(\eta)=\frac{1}{k!}\left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p\,=\,0}(k>0), \tag{3.2.14}$$

$$\psi_0(\eta)=G(\eta,0)=g_0(\eta),$$

$$\psi_k(\eta)=\frac{1}{k!}\left.\frac{\partial^k G(\eta,p)}{\partial p^k}\right|_{p\,=\,0}(k>0). \tag{3.2.15}$$

We have due to (3.2.12) - (3.2.13) that

$$F(\eta,p)=\phi_0(\eta)+\sum_{k=1}^{+\infty}\phi_k(\eta)p^k, \tag{3.2.16}$$

$$G(\eta,p)=\psi_0(\eta)+\sum_{k=1}^{+\infty}\psi_k(\eta)p^k. \tag{3.2.17}$$

The convergence region of the above series depends upon the linear operator L and the non-zero parameter h which is to be selected such that solution converges at $p=1$. Using equation (3.2.14) for $p=1$, we get

$$f(\eta)=\phi_0(\eta)+\sum_{m=1}^{+\infty}\phi_m(\eta), \tag{3.2.18}$$

$$g(\eta)=\psi_0(\eta)+\sum_{m=1}^{+\infty}\psi_m(\eta). \tag{3.2.19}$$

Here ϕ_m and ψ_m are unknowns to be determined.

Differentiating m times the two sides of equations (3.2.2) about the embedding parameter p, using Leibnitz theorem, setting $p = 0$ and dividing by $m!$, we get

$$L[\phi_m - \chi_m \phi_{m-1}] = hR_m(\eta), \tag{3.2.20}$$

where

$$\chi_m = \begin{cases} 0 & \text{when } m \leq 1 \\ 1 & \text{when } m > 1 \end{cases}, \tag{3.2.21}$$

$$R_m(\eta) = \phi_{m-1}{}'''(\eta) + \sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\phi_k{}''(\eta) - \sum_{k=0}^{m-1} \phi'_{m-1-k}(\eta)\phi'_k(\eta) - M^2\phi'_{m-1}(\eta), \tag{3.2.22}$$

with boundary conditions

$$\phi_m(0) = \phi'_m(0) = \phi'_m(+\infty) = 0. \tag{3.2.23}$$

Similarly differentiating m times both sides of equations (3.2.3) about the embedding parameter p, using Leibnitz theorem, setting $p = 0$ and dividing by $m!$, we get

$$L[\psi_m - \chi_m \psi_{m-1}] = hW_m(\eta), \tag{3.2.24}$$

where

$$\chi_m = \begin{cases} 0 & \text{when } m \leq 1 \\ 1 & \text{when } m > 1 \end{cases}, \tag{3.2.25}$$

$$W_m(\eta) = \psi''_{m-1}(\eta) + Pr\sum_{k=0}^{m-1} \phi_{m-1-k}(\eta)\psi'_k(\eta) - Pr\lambda\psi'_m(\eta), \tag{3.2.26}$$

with boundary conditions

$$\psi_m(0)\text{=}0, \psi_m(+\infty)\text{=}0. \tag{3.2.27}$$

From equations (3.2.20) we get equations in terms of $\phi_m(\eta)$, solving them we get $\phi_0, \phi_1, \phi_2, \phi_3,$ ... Thus

$$f(\eta) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + ... \tag{3.2.28}$$

From equations (3.2.24) we get equations in terms of $\psi_m(\eta)(m \geq 1)$, solving them we get ψ_1, ψ_2, ψ_3 ...

We have

$$g(\eta) = \psi_0 + \psi_1 + \psi_2 + \psi_3 + ... \tag{3.2.29}$$

The solutions f and g consists of h and is a series solution. To get a valid solution we have to choose h in such a way that both series are convergent. We can generate large number of terms on solving the linear equations by MATHEMATICA.

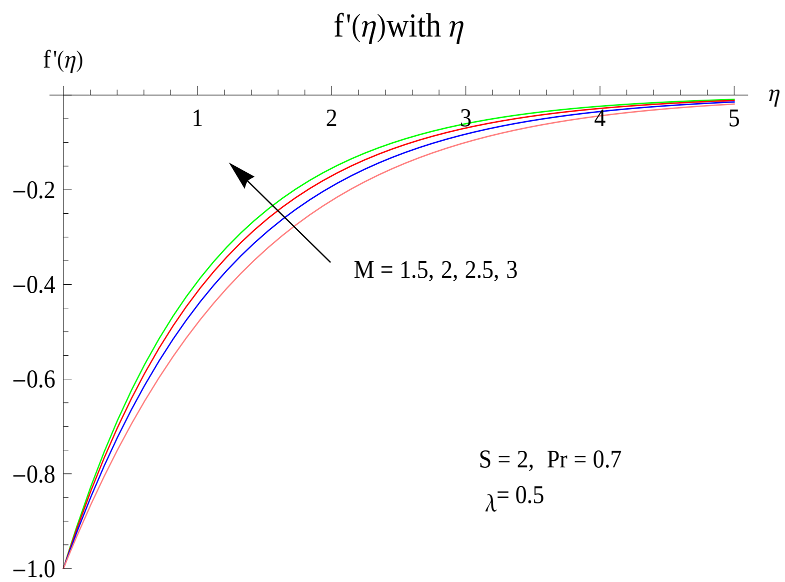

Figure 3.1:Velocity Profiles for several values of M

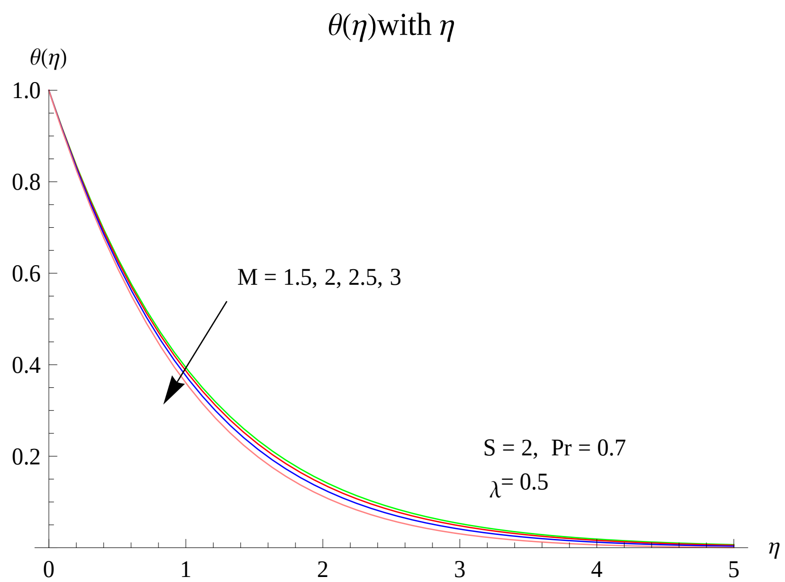

Figure 3.2:Temperature Profiles for several values of M

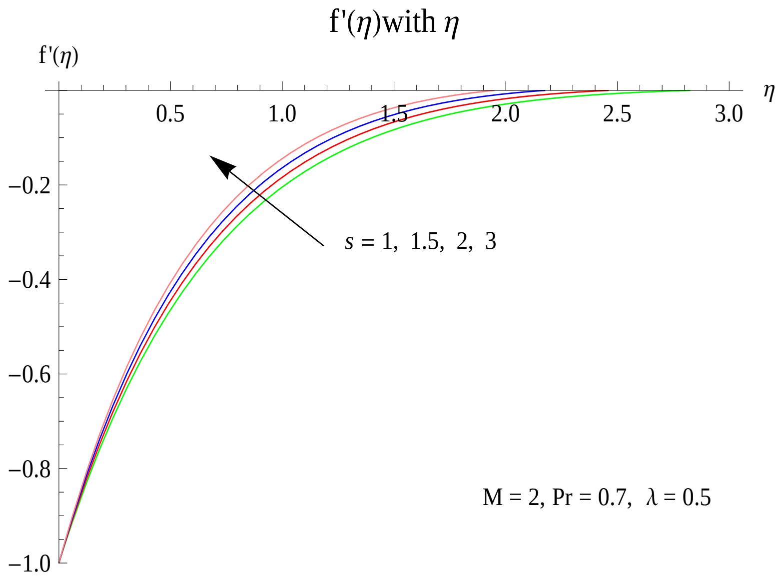

Figure 3.3:Velocity Profiles for several values of s

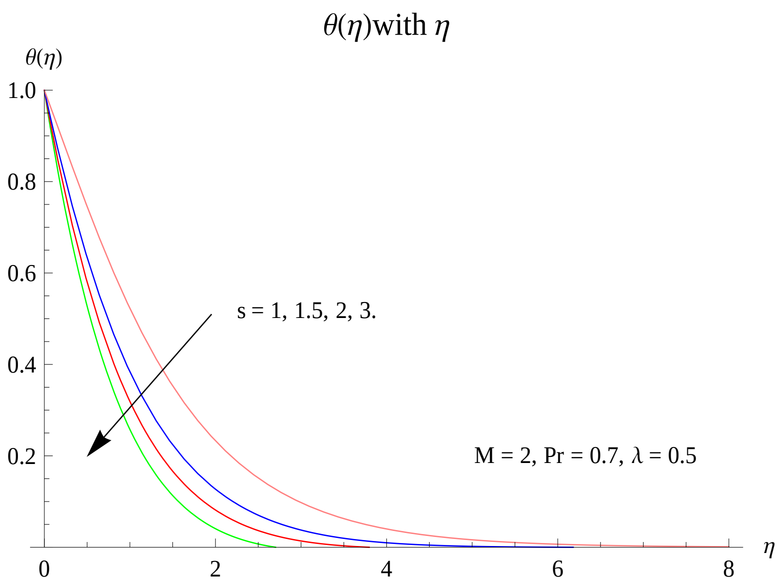

Figure 3.4:Temperature Profiles for several values of s

3.3 Result and Discussions

Figure 3.1 represent velocity curves plotted $f'(\eta)$ obtained from homotopy analysis method (HAM) with reference to η for different values of M = 1.5, 2, 2.5, 3.0. with s = 2, Pr = 0.7, λ = 0.5.

Figure 3.2 represent Temperature profiles plotted $\theta(\eta)$ with reference to η fdifferent values of M = 1.5, 2, 2.5, 3.0. with s = 2, Pr = 0.7, λ = 0.5.

Figure 3.3 represent velocity curves plotted $f'(\eta)$ obtained from homotopy analysis method (HAM) with reference to η for different values of s = 1, 1.5, 2, 3.0. with M = 2, Pr = 0.7, λ = 0.5.

Figure 3.4 represent Temperature profiles plotted $\theta(\eta)$ with reference to η different values of s = 1, 1.5, 2, 3.0. with M = 2, Pr = 0.7, λ = 0.5.

The solution obtained contains exponential of negative powers of unknown variable so the solution is convergent.

Figures 3.1, 3.2, 3.3, 3.4 match with the solutions obtained by [8]

Thus we conclude that the solutions obtained by HAM coincide with numerical solution obtained by Krishnendu Bhattacharyya [8].

We can use homotopy analysis method which gives good accurate solution to boundary value problems and may be treated as strong analytic method to solve non linear differential equations.

Chapter 4

Analytical Solution of Unsteady Laminar Incompressible Flow in the Presence of Transverse Magnetic Field

4.1 Introduction

Magneto hydrodynamics stagnation point flows are relevant to many engineering applications such as Metallurgy Industries, MHD pumps, Heat Exchangers, Petroleum Engineering. Many scientists have studied such problems.

4.2 Mathematical formulation of the problem

The unsteady, laminar incompressible flow of a viscous fluid in the presence of transverse magnetic field near the stagnation point of a flat sheet coinciding with the plane y = 0 , the flow being confined to positive direction of y - axis. Initially the surface is at rest in an unbounded quiescent fluid with uniform temperature . Magnetic field is applied along

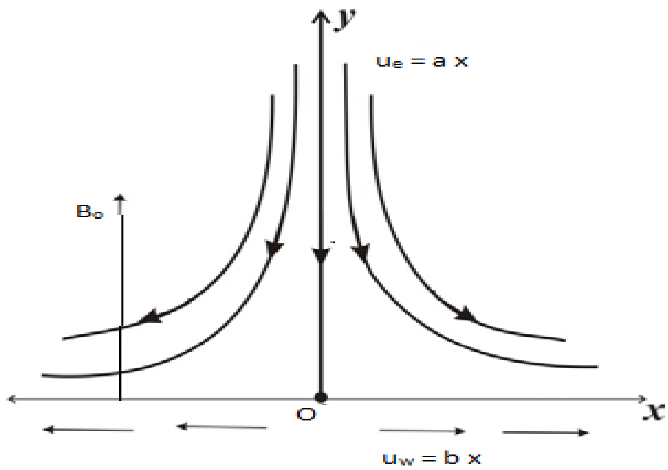

Figure 4.1 : Flow Configuration

positive direction of x - axis.

At time $t > 0$, the surface is suddenly stretched with the local tangential velocity $u_w = bx$ (b is a positive constant) keeping the origin fixed and x- coordinate measured along the stretching surface from the stagnation point O.

For $t > 0$, The velocity distribution in the potential flow (free stream velocity), given by $u_e = ax$ (a is a positive constant), starts impulsively in motion from rest. Flow becomes unsteady due to the impulsive motion of sheet.

u and v are velocity components along x and y - directions, respectively; T is the temperature;The unsteady boundary layer equations can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = a^2 x + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2 u}{\rho} \tag{4.2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{4.2.3}$$

with the initial and boundary conditions:

$$\mathrm{t} < 0,\; u(\mathrm{x,y,t}) = \mathrm{v}(x,\mathrm{y,t}) = 0,\;\; \mathrm{T}(x,\mathrm{y,t}) = T_\infty,\; \text{ for all } x, y \tag{4.2.4}$$

$$\mathrm{t} \geq 0,\; u(\mathrm{x,y,t}) = u_w\text{=bx,}\mathrm{v}(x,\mathrm{y,t}) = 0,\;\; \mathrm{T}(x,\mathrm{y,t}) = T_w,\; \text{ for } y = 0 \tag{4.2.5}$$

$$u(\mathrm{x,y,t}) = u_e\text{=ax,}\mathrm{T}(x,\mathrm{y,t}) = T_w, \text{for y} \to \infty. \tag{4.2.6}$$

ν and α denote kinematic viscosity and thermal diffusivity respectively. The subscripts e, w, and ∞ denote the conditions at the edge of the boundary-layer, on the wall and in the free stream, respectively.

The stream function ψ and the similarity transformations are taken as

$\eta = \left(\frac{b}{\nu}\right)^{1/2} \xi^{-1/2} y$, $\xi = 1 - e^{-bt}$, $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, $\psi(x,y,t) = (b\nu)^{1/2} \xi^{1/2} x f(\eta, \xi)$,

$u = bxf'(\eta, \xi)$,

$v = -(b\nu)^{1/2} \xi^{1/2} f(\eta, \xi)$, $\mathrm{Pr} = \frac{\nu}{\alpha}$, $T = T_\infty + (T_w - T_\infty) G(\eta, \xi)$.

Equations (4.2.1) - (4.2.3) reduce to the system of non linear partial differential equations

$$\begin{aligned} &\frac{\partial^3 f}{\partial \eta^3} + \left(\frac{\eta}{2}\right)(1-\xi)\frac{\partial^2 f}{\partial \eta^2} + \xi\left(\lambda^2 + f\frac{\partial^2 f}{\partial \eta^2} - \left(\frac{\partial f}{\partial \eta}\right)^2\right) \\ &-\frac{\sigma B_0^2}{\rho}\left(\frac{\partial f}{\partial \eta}\right) - \xi(1-\xi)\frac{\partial^2 f}{\partial \eta \partial \xi} = 0, \end{aligned} \tag{4.2.7}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 G}{\partial \eta^2} + \left(\frac{\eta}{2}\right)(1-\xi)\frac{\partial G}{\partial \eta} - \xi(1-\xi)\frac{\partial G}{\partial \eta} + f\frac{\partial G}{\partial \eta} = 0 \tag{4.2.8}$$

subject to boundary conditions

$$f(0,\xi) = 0, f'(0,\xi) = 1, f'(\infty,\xi) = \lambda \tag{4.2.9}$$

$$\mathrm{G}(0,\xi) = 1, G(\infty,\xi) = 0 \tag{4.2.10}$$

Here $\lambda = a/b$ is a positive constant denoting velocity ratio parameter; η and ξ are the transformed dimensionless independent variables; ψ is the stream function; f is the dimensionless stream function; f' is the dimensionless velocity; G is dimensionless temperature and Pr is the Prandtl number.

The study is first to present application of HAM to an unsteady flow

The two special cases are

initial steady state : ξ = 0

final unsteady state : ξ = 1

Equations (7) to (10), for unsteady flow reduce to

$$\frac{\partial^3 f}{\partial \eta^3} + \left(\frac{\eta}{2}\right)\frac{\partial^2 f}{\partial \eta^2} - M\left(\frac{\partial f}{\partial \eta}\right) = 0 \tag{4.2.11}$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2 G}{\partial \eta^2} + \left(\frac{\eta}{2}\right)\frac{\partial G}{\partial \eta} = 0 \tag{4.2.12}$$

$$f(0,0) = 0,\, f'(0,0) = 1,\, f'(\infty,0)=\lambda \tag{4.2.13}$$

$$G(0,0)=1, G(\infty,0)=0 \tag{4.2.14}$$

4.3 Homotopy Analysis Method (HAM)

To apply the homotopy analysis method to the problem (4.2.11) under boundary condition (4.2.13), we first select the auxiliary linear operator as

$$L = \frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2}, \tag{4.3.1}$$

Then we construct a family of partial differential equations

$$(1-p)L\left[F(\eta,p) - f_0(\eta)\right] = hp\left\{\frac{\partial^3 F}{\partial \eta^3} + \frac{\eta}{2}\frac{\partial^2 F}{\partial \eta^2} - M\frac{\partial F}{\partial \eta}\right\} \tag{4.3.2}$$

with boundary conditions

$$F(0,p) = 0, F_\eta(0,p) = 1, F_\eta(+\infty,p) = 0, \tag{4.3.3}$$

where subscript η denotes the first-order derivative with respect to η , $p \in [0,1]$ is the embedding parameter, h is an non zero auxiliary parameter. We choose the initial guess $f_0(\eta)$ as follows in accordance with boundary conditions (4.2.13) as

$$f_0(\eta) = (1-\lambda) + \lambda\eta + (\lambda-1)e^{-\eta}, \tag{4.3.4}$$

When p = 0, we have the solution

$$F(\eta,0) = f_0(\eta), \tag{4.3.5}$$

when p = 1, equation (4.3.2) is the same as (4.2.11) , so that

$$F(\eta,1) = f(\eta), \tag{4.3.6}$$

Thus as p increases from 0 to 1, the solution varies from the initial guess $f_0(\eta)$ to the exact solution $f(\eta)$. The initial guess approximation $f_0(\eta)$, the linear operator L and the parameter h are to be selected such that the equation (4.3.2) has solution at each point $p \in [0,1]$ and also $F(\eta,p)$ can be expressed in Maclaurin series as

$$F(\eta,p) = F(\eta,0) + \sum_{k=1}^{+\infty} \frac{p^k}{k!} \left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p=0}, \tag{4.3.7}$$

Defining

$$\phi_0(\eta) = F(\eta,0) = f_0(\eta)$$

,

$$\phi_k(\eta) = \frac{1}{k!}\left.\frac{\partial^k F(\eta,p)}{\partial p^k}\right|_{p=0}(k>0), \tag{4.3.8}$$

Equation (4.3.7) becomes

$$F(\eta,p) = \phi_0(\eta) + \sum_{k=1}^{\infty}\phi_k(\eta)p^k. \tag{4.3.9}$$

The convergence region of the above series depends upon the linear operator L and the non-zero parameter h which are to be selected such that solution converges at p = 1. Using p = 1 in equations (4.3.9) , we get

$$f(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty}\phi_m(\eta), \tag{4.3.10}$$

where $\phi_m(\eta)$ are the unknowns to be determined.

Differentiating equation (4.3.2) m times about the embedding parameter p, using Leibnitz theorem, setting p = 0 and dividing by m!, we get

$$L\left[\phi_m - \chi_m\phi_{m-1}\right] = hR_m(\eta), \tag{4.3.11}$$

where

$$\chi_m = \begin{cases} 0\ when\ m \leq 1 \\ 1\ when\ m > 1, \end{cases} \tag{4.3.12}$$

$$R_m[\eta] = \phi_{m-1}^{'''} + \left(\frac{\eta}{2}\right)\phi_{m-1}^{''} - M\phi_{m-1}^{'}, \tag{4.3.13}$$

with boundary conditions

$$\phi_m(0) = \phi_m^{'}(0) = \phi_m^{'}(\infty) = 0. \tag{4.3.14}$$

$$f(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty}\phi_m(\eta), \tag{4.3.15}$$

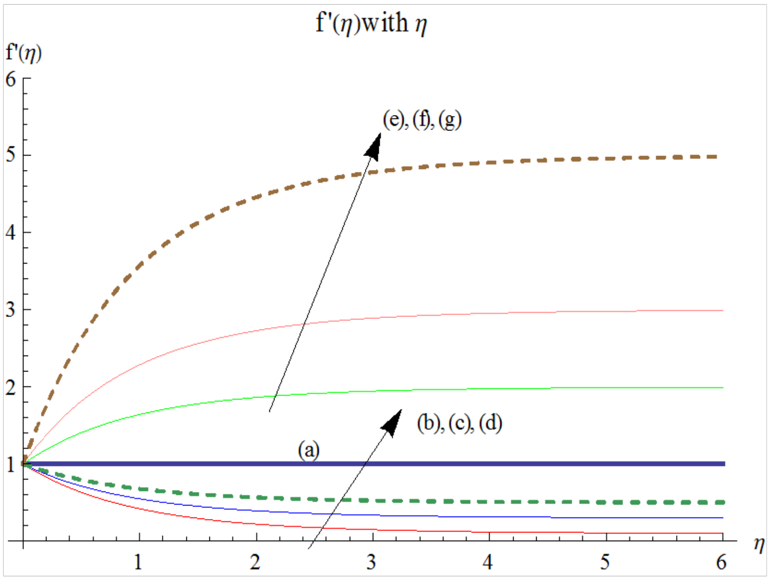

Figure 4.2 : M = 0,
$(a)\lambda = 1, (b)\lambda = 0.1, (c)\lambda = 0.3,$
$(d)\lambda = 0.5, (e)\lambda = 2, (f)\lambda = 3, (g)\lambda = 5.$

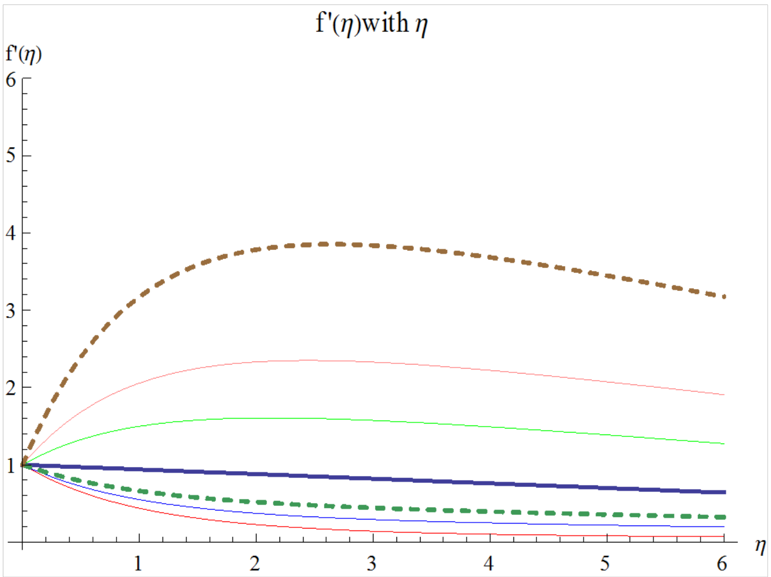

Figure 4.3: M = 0.5
$(a)\lambda = 1, (b)\lambda = 0.1, (c)\lambda = 0.3,$
$(d)\lambda = 0.5, (e)\lambda = 2, (f)\lambda = 3, (g)\lambda = 5.$

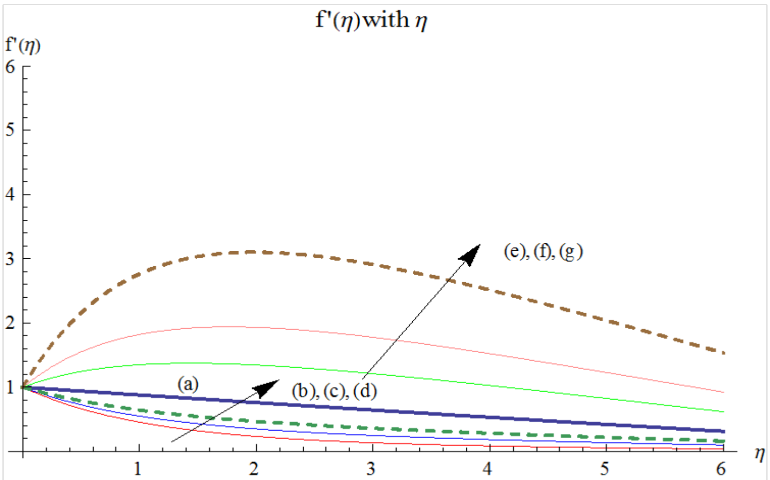

Figure 4.4: M = 1
$(a)\lambda = 1, (b)\lambda = 0.1, (c)\lambda = 0.3,$
$(d)\lambda = 0.5, (e)\lambda = 2, (f)\lambda = 3, (g)\lambda = 5.$

4.4 Results and Discussions

Graphs are plotted $f'(\eta)$ versus η for different values of velocity ratio parameter λ, and different values of M .

Figure 4.2 represent velocity curves plotted $f'(\eta)$ with reference to η for different values of λ = 1, 0.1, 0.3, 0.5, 2, 3. with M = 0.

Figure 4.3 represent velocity curves plotted $f'(\eta)$ with reference to η for different values of λ = 1, 0.1, 0.3, 0.5, 2, 3. with M = 0.5

Figure 4.4 represent velocity curves plotted $f'(\eta)$ with reference to η for different values of λ = 1, 0.1, 0.3, 0.5, 2, 3. with M = 1.

Boundary layer thickness is considerably reduced with the effect of MHD. This is a important discovery and will contribute to Boundary layer theory.

Chapter 5

Summary, Conferences - Paper Presented/Attended and Reference

5.1 Summary

This chapter highlights the Analytical Method of solving non linear boundary value problems, i.e., the importance of Homotopy Analysis Method and the plan of future work.

In Chapter 1 basic definitions of fluid mechanics,Boundary layer theory, mathematical methods, Homotopy analysis method, review of literature, need and scope of research is discussed.

Chapter 2 deals with introduction to Homotopy analysis method. The illustration of the method is dealt in detail by an example.

Chapter 3 deals with problems discussed by Krishnendu Bhattacharyya [8] . Homotopy analysis method is used to solve the problem and using the solution obtained graphs are drawn. The results obtained are compared with the results obtained in [8] by numerical methods which confirms that Homotopy analysis method is a efficient method .

In Chapter 4 The study of unsteady, laminar incompressible flow of a viscous fluid in the presence of transverse magnetic field near the stagnation point of a flat sheet coinciding with the plane y = 0 , the flow being confined to positive direction of y - axis is discussed.

The important observation is Boundary layer thickness considerably reduces with the effect of Magnetic Field. This result is a important discovery and will contribute to Boundary layer theory.

Conferences - Paper Presented/Attended

- Presented a paper " Analytical solution of a flow of a Navier Stokes fluid due to stretching Boundary " at International Conference on Mathematical modeling and Non linear equations organized Department of Mathematics; B N M Institute of Technology , Bangalore on 20 - 22 January 2010.

- Presented a paper "Homotopy analysis method applied to stretching sheet problems" at National Conference on "Emerging Trends in Fluid Mechanics and Graph Theory " at Christ University, Bangalore on February 25 and 26, 2010.

- Presented a paper "A New Analytical Solution To Boundary Layer Problem" at II National Conference on "Emerging Trends in Fluid Mechanics and Graph Theory " at Christ University, Bangalore on February 11 and 12, 2011.

- Presented as a paper "Strong Approxmate Analytic Solution Of A Boundary Layer Problem " at International Conference on "Mathematical Modelling And Applications To Industrial Problems (MMIP2011)" at National Institute of Technology Calicut, Kerala, India, during March 28-31, 2011.

- Presented a paper ”Homotopy Analysis Method of A Boundary Layer Problem ” at National Conference on ”Emerging Trends In Information Technology And Mathematics (ETITM 2011)” at East West Institute of Technology , Bangalore, India during November 3 - 4 , 2011.

- Presented a paper ”Homotopy Analysis Solution for MHD Sink Flow ” at National Conference on ”Frontiers in Applied Mathematics” organized by Research Centre in Applied Mathematics, MES College, Malleswaram, Bangalore on 9th and 10th March 2012.

- Presented a paper ”Boundary Layer Problem in MHD And Sink Flow” at National Conference on DOCAM’ 2012 during May 18 - 19, 2012 at East West Institute of Technology, Bangalore, India.

- Attended an ” International Conference on Fluid Mechanics and Graph Theory ”, ICFGD 2012, held on 16th and 18th August, 2012 at Christ University, Bangalore.

- Presented a paper ”Approximate Analytical Solution For Compressible Boundary Layer Problem” at 57th Congress of ISTAM (An International Meet), held on 17th - 20th December, 2012 at Defence Institute of Advanced Technology, Pune.

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