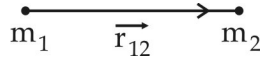


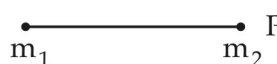
GRAVITATION (For V SEMESTER)

- (a) **Newton's law of gravitation:** Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres.


 Force on m_2 due to m_1 in vector form is $\vec{F}_{21} = -\frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12}$.

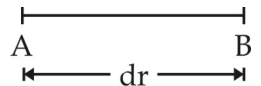
In general $F = \frac{Gm_1m_2}{r^2}$.

- (b) **Gravitational Field:** Gravitational field intensity at a point is defined as gravitational force experienced by a unit mass kept at that point.


 $\vec{E} = \frac{F}{m_2}$ [$\vec{E} \rightarrow$ intensity at P on m_2]

$F = -\frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12} \Rightarrow E = -\frac{Gm_1}{r_{12}^2} \hat{r}_{12}$

- (c) **Gravitational potential:** Gravitational potential at a point is the potential energy that a unit mass would possess at the point in the gravitational field.


 Potential difference (dV) between points A and B $dV = -E dr$.

Gravitational intensity $E = -\frac{dV}{dr}$.

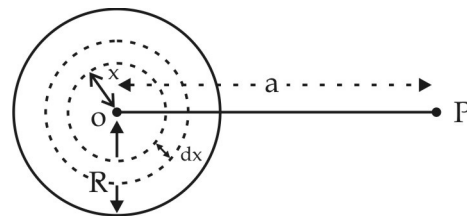
Also gravitational field is the negative potential gradient at a point is $V = -\frac{Gm}{r}$.

Gravitational potential energy of a body at a point, $PE = -\frac{Gm_1m_2}{r}$.

Expression for gravitational potential and field at a point due to a solid sphere:

- (a) **Point P outside the sphere:**

Consider a solid sphere of mass M , radius R and density ρ . Consider it to be made of large number of thin spherical shells whose radii vary from O to R . Let x be radius of one such shell and dx its thickness.



Mass of the shell $= (4\pi x^2 dx)\rho$

$(m = V\rho$ where $V = \text{Area} \times \text{thickness} = 4\pi x^2 \times dx)$

The gravitational potential at point P due to shell is

$dV = -\frac{Gm}{a} = -\frac{G(4\pi x^2 dx)\rho}{a} = -\frac{G4\pi\rho}{a} x^2 dx$

The potential due to the entire sphere at P is

$$V = -\frac{G4\pi\rho}{a} \int_0^R x^2 dx = -\frac{G4\pi\rho}{a} \left[\frac{x^3}{3} \right]_0^R$$

$$V = -\frac{G4\pi\rho}{a} \left[\frac{x^3}{3} \right]_0^R = -\frac{G4\pi\rho}{3a} R^3$$

$$V = -\frac{G}{a} \left[\frac{4}{3} \pi R^3 \rho \right] \text{ or } \boxed{V = -\frac{GM}{a}} \text{ where } M = \frac{4}{3} \pi R^3 \rho$$

Gravitational field:

$$E = -\frac{dV}{dr} = -\frac{dV}{da} = -\frac{d}{da} \left[-\frac{GM}{a} \right] = -\frac{GM}{a^2}$$

$$\boxed{E = -\frac{GM}{a^2}}$$

(b) Point P on the surface of the sphere:

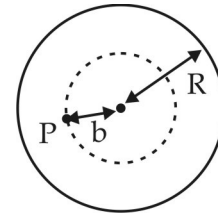
$$V = -\frac{GM}{R} [\because a = R] \text{ and } E = -\frac{GM}{R^2} [a^2 = R^2]$$

(c) Point P inside the sphere:

Let P be a point inside the sphere of radius R on a imaginary spherical surface of radius b. The gravitational potential at P due to all shells of radii between R and b (outside point P) s

$$V_1 = -\int_b^R \frac{G(4\pi x^2 dx \rho)}{x} \quad \text{mass of a shell} = 4\pi x^2 dx \rho$$

$$V_1 = -G4\pi\rho \int_b^R x dx = -G4\pi\rho \left[\frac{x^2}{2} \right]_b^R = -\frac{G4\pi\rho}{2} [R^2 - b^2] \text{ ----- (1)}$$



The potential at P due to shells within the sphere of radius b is

$$V_2 = -\frac{G}{b} \left[\frac{4}{3} \pi b^3 \rho \right] \text{ ----- (2)} \quad \left[\frac{4}{3} \pi b^3 \rho = m = \text{mass of sphere of radius } b \right]$$

The potential at P due to entire sphere is

$$V = V_1 + V_2 = -\frac{G4\pi\rho}{2} [R^2 - b^2] - \frac{4}{3} G\pi\rho b^2$$

$$V = -G4\pi\rho \left[\frac{R^2 - b^2}{2} + \frac{b^2}{3} \right] = -G4\pi\rho \left[\frac{R^2}{2} - \frac{b^2}{2} + \frac{b^2}{3} \right]$$

$$V = -G4\pi\rho \left[\frac{3R^2 - b^2}{6} \right] \text{ or } V = -\frac{G4\pi\rho}{6} [3R^2 - b^2] \text{ ----- (3)}$$

$$\text{or } V = -\frac{G}{2R^3} \left[\frac{4}{3} \pi R^3 \rho \right] (3R^2 - b^2) \text{ [by multiplying and dividing equation (3) by } R^2]$$

$$\boxed{V = -\frac{GM}{2R^3} [3R^2 - b^2]} \quad \left[\because M = \frac{4}{3} \pi R^3 \rho \right]$$

Gravitational field:

$$E = -\frac{dV}{db} = -\frac{d}{db} \left[-\frac{GM}{2R^3} (3R^2 - b^2) \right]$$

$$E = \frac{GM}{2R^3} \frac{d}{db} (3R^2 - b^2) = \frac{GM}{2R^3} (-2b) \quad \text{or} \quad \boxed{E = -\frac{GMb}{R^3}}$$

Kepler's Laws of Planetary Motion:

I law - Law of orbits: *Every planet moves in an elliptical orbit with sun being at one of its foci.*

II Law - Law of areas: *The radius vector drawn from the sun to the planet sweeps equal areas in equal intervals of time. i.e., the area velocity is constant.*

III Law - Law of periods: *The square of the time period of revolution of a planet around the sun is proportional to the cube of the semi - major axis of the elliptical orbit.*

Derivation of Kepler's Second Law:

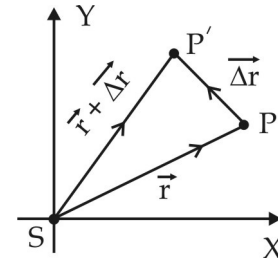
From Newton's Law of gravitation

$$\vec{F} = -\frac{GMm}{r^2} \hat{r} \quad \text{----- (1)}$$

$M \rightarrow$ mass of sun

$m \rightarrow$ mass of planet

$r \rightarrow$ distance between them



The force experienced by the planet is $\vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt}$ ----- (2)

From (1) and (2) $m \frac{d\vec{v}}{dt} = -\frac{GMm}{r^2} \hat{r}$ as $\vec{r} = r\hat{r} \Rightarrow \hat{r} = \frac{\vec{r}}{r}$.

$$\therefore \frac{d\vec{v}}{dt} = -\frac{GM}{r^3} \vec{r}$$

Taking left cross product on both sides of this equation with \vec{r} .

$$\vec{r} \times \frac{d\vec{v}}{dt} = -\frac{GM}{r^3} (\vec{r} \times \vec{r}) \quad \text{or} \quad \frac{d}{dt} (\vec{r} \times \vec{v}) = -\frac{GM}{r^3} (\vec{r} \times \vec{r})$$

As $\vec{r} \times \vec{r} = 0 \quad \therefore \frac{d}{dt} (\vec{r} \times \vec{v}) = 0 \quad \text{or} \quad \vec{r} \times \vec{v} = \text{constant}$

Also $\vec{r} \times \frac{d\vec{r}}{dt} = \text{constant} \quad \left[\because \vec{v} = \frac{d\vec{r}}{dt} \right]$

Multiplying and dividing LHS by 2, we get

$$2 \left[\frac{1}{2} \left(\vec{r} \times \frac{d\vec{r}}{dt} \right) \right] = \text{constant} \quad \text{----- (1)}$$

In a interval of time Δt , the planet covers an area ΔA , then from the diagram, area SPP' is $\Delta \vec{A} = \frac{1}{2}(\vec{r} \times \Delta \vec{r})$.

The area velocity of the planet $\vec{h} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{A}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1}{2} \left(\vec{r} \times \frac{\Delta \vec{r}}{\Delta t} \right)$ or $\vec{h} = \frac{1}{2} \left[\vec{r} \times \frac{d\vec{r}}{dt} \right]$ ----- (2) or $\vec{h} = \frac{1}{2} [\vec{r} \times \vec{v}]$

Comparing (1) and (2) we get $h = \frac{1}{2} \times \text{constant}$ or $h = \text{constant}$.

Thus area velocity of the planet is a constant. Hence IInd law.

Derivation of Kepler's First Law:

From second law, areal velocity = constant

or $\vec{h} = \frac{1}{2}(\vec{r} \times \vec{v}) \Rightarrow 2\vec{h} = (\vec{r} \times \vec{v})$

As $\vec{r} = r \hat{r} \quad \therefore 2\vec{h} = r \hat{r} + \frac{d}{dt}(r \cdot \hat{r}) \quad \left[\because \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r \cdot \hat{r}) \right]$

$2\vec{h} = r \hat{r} \times \left[r \frac{d\hat{r}}{dt} + \frac{dr}{dt} \hat{r} \right]$

$2\vec{h} = r^2 \hat{r} \times \frac{d\hat{r}}{dt} + r \frac{dr}{dt} (\hat{r} \times \hat{r})$

$2\vec{h} = r^2 \hat{r} \times \frac{d\hat{r}}{dt} \quad [\because \hat{r} \times \hat{r} = 0] \quad \text{or} \quad h = \frac{r^2}{2} \left(\hat{r} \times \frac{d\hat{r}}{dt} \right)$ ----- (1)

Taking right cross product on both sides of (2) with \vec{h}

$\left[\frac{d\vec{v}}{dt} = -\frac{GM}{r^2} \hat{r} \text{----- (2)} \right] \Rightarrow \frac{d\vec{v}}{dt} \times \vec{h} = -\frac{GM}{r^2} \hat{r} \times \vec{h}$ ----- (3)

Substituting for \vec{h} from (1) in RHS of equation (3)

$\frac{d\vec{v}}{dt} \times \vec{h} = -\frac{GM}{r^2} \hat{r} \times \frac{r^2}{2} \left(\hat{r} \times \frac{d\hat{r}}{dt} \right) = -\frac{GM}{2} \left[\hat{r} \times \left(\hat{r} \times \frac{d\hat{r}}{dt} \right) \right]$ ----- (4)

From vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$\therefore \vec{r} \times \left(\vec{r} \times \frac{d\vec{r}}{dt} \right) = \left(\vec{r} \cdot \frac{d\vec{r}}{dt} \right) \vec{r} - (\vec{r} \cdot \vec{r}) \frac{d\vec{r}}{dt}$

As \hat{r} is perpendicular to $\frac{d\vec{r}}{dt}$ (velocity) $\therefore \vec{r} \cdot \frac{d\vec{r}}{dt} = 0, \hat{r} \cdot \hat{r} = 1$

$\therefore \vec{r} \times \left(\vec{r} \times \frac{d\vec{r}}{dt} \right) = -\frac{d\vec{r}}{dt}$

Thus equation (4) becomes $\frac{d\vec{v}}{dt} \times \vec{h} = -\frac{GM}{2} \left[-\frac{d\hat{r}}{dt} \right]$

or $\frac{d\vec{v}}{dt} \times \vec{h} = \frac{GM}{2} \frac{d\hat{r}}{dt}$ or $\frac{d}{dt}(\vec{v} \times \vec{h}) = \frac{GM}{2} \frac{d\hat{r}}{dt}$ ----- (5)

Integrating equation (5)

$$\vec{v} \times \vec{h} = \frac{GM}{2} \hat{r} + \vec{c}$$

Taking dot - product on both sides of the above equation with \vec{r} , we get,

$$\begin{aligned} \vec{r} \cdot (\vec{v} \times \vec{h}) &= \frac{GM}{2} \vec{r} \cdot \hat{r} + \vec{r} \cdot \vec{c} \\ &= \frac{GM}{2} r \hat{r} \cdot \hat{r} + rc \cos \theta \end{aligned}$$

$$\vec{r} \cdot (\vec{v} \times \vec{h}) = \frac{GM}{2} r + rc \cos \theta \text{ ----- (6)}$$

using vector identity $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

$$\begin{aligned} \vec{r} \cdot (\vec{v} \times \vec{h}) &= \vec{h} \cdot (\vec{r} \times \vec{v}) \\ &= \vec{h} \cdot (2\vec{h}) \quad [\because 2\vec{h} = \vec{r} \times \vec{v}] \\ &= 2h^2 \end{aligned}$$

Thus equation (6) can be written as

$$2h^2 = \frac{GM}{2} r + rc \cos \theta = r \left[\frac{GM}{2} + c \cos \theta \right] \text{ or } \frac{1}{r} = \frac{GM}{4h^2} + \frac{c \cos \theta}{2h^2}$$

Multiplying throughout by $\frac{4h^2}{GM}$

$$\frac{4h^2}{GM r} = 1 + \left(\frac{2C}{M} \right) \cos \theta \text{ let } l = \frac{4h^2}{GM}, e = \frac{2c}{GM}$$

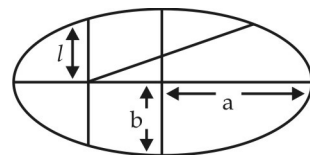
Then $\boxed{\frac{l}{r} = 1 + e \cos \theta}$ ----- (7)

This is the equation of a conic section with $l \rightarrow$ semi latus rectum and $e \rightarrow$ eccentricity. If $\theta < 1$, then the above equation is the equation of ellipse with sun at one of its foci. This proves Kepler's first law.

Derivation of Kepler's third law:

The path of the planet with sun at one of the foci is as shown

$a \rightarrow$ semi major axis, $b \rightarrow$ semi minor axis, $l \rightarrow$ semi latus rectum of the ellipse.



For an ellipse $b^2 = a^2(1 - e^2)$ (1) and $l = a(1 - e^2)$ ----- (2)

Dividing (2) / (1) $\frac{l}{b^2} = \frac{1}{a}$ or $l = \frac{b^2}{a}$

$$\text{As } l = \frac{4h^2}{GM} \Rightarrow l = \frac{4h^2}{GM} = \frac{b^2}{a} \quad \text{or} \quad b^2 = \frac{4h^2 a}{GM} \text{ ----- (3)}$$

The period of revolution of the planet round the sun is T is given by $T = \frac{\pi ab}{h}$.

$$\begin{aligned} [\because \text{areal velocity}] \quad h &= \frac{\text{area of ellipse}}{\text{period of revolution}} \\ &= \frac{\pi ab}{T} \end{aligned}$$

$$\text{or} \quad T^2 = \frac{\pi^2 a^2 b^2}{h^2} \text{ ----- (4)}$$

substituting for b^2 from (3) in (4)

$$T^2 = \frac{\pi^2 a^2}{h^2} \left[\frac{4h^2 a}{GM} \right] = \frac{4\pi^2 a^3}{GM}$$

$$\text{or} \quad \frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{constant} \quad \text{or} \quad \boxed{T^2 \propto a^3}$$

Hence the Kepler's third law.

Inertial mass and gravitational mass:

If a force \vec{F} acts on a body of mass m , and the acceleration of the body is \vec{a} , then from Newton's second law $\vec{F} = m_i \vec{a}$, where m_i is called the inertial mass.

This body is also acted upon by gravitational force. Then the mass of the body is called gravitational mass m_g . The gravitational force on the body by the earth is

$$\vec{F} = \frac{GM_e m_g}{R_e^2} \text{ ----- (1)}$$

$$\text{From Newton's second law } F = m_i g \text{ ----- (2)}$$

$$\text{Comparing (1) and (2)} \quad m_i g = \frac{GM_e m_g}{R_e^2} \quad \text{or} \quad \frac{m_i}{m_g} = \frac{GM_e}{gR_e^2} \text{ ----- (3)}$$

At a place, RHS of above equation is a constant.

$$\therefore \text{ratio } \frac{m_i}{m_g} = K = \text{constant}$$

If $K=1$, then $G=6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$. For any other value of K , G is different. Thus $m_i = m_g$.

Thus inertial mass and gravitational mass are equivalent.

Satellite motion: Objects that move around a planet is called a satellite. Moon is a natural satellite of earth. Man made satellite that revolve around the planets are called artificial satellites.

Elements of satellite motion:

- Orbital velocity:** The velocity with which a satellite revolves round a planet is called orbital velocity. Consider a satellite of mass m , moving around the earth in an orbit of radius r . Let M be the mass of the earth and v_o be the orbital velocity of the satellite. The necessary centripetal force acting on the satellite is provided by the gravitational force of attraction on the satellite.

$$\text{i.e., } F_c = F_g \quad \left[F_c = \frac{mv_o^2}{r} \quad F_g = \frac{GMm}{r^2} \right]$$

$$\frac{mv_o^2}{r} = \frac{GMm}{r^2} \Rightarrow v_o^2 = \frac{GM}{r} \quad \text{or} \quad \boxed{v_o = \sqrt{\frac{GM}{r}}} \text{----- (1)}$$

If $g \rightarrow$ acceleration due to gravity on the satellite near the surface of earth, then $mg = \frac{GMm}{R^2}$.

or $gR^2 = GM$ ----- (2)

Substituting for GM from (2) in (1), $v_o = \sqrt{\frac{gR^2}{r}}$

If the satellite is close to earth, then $r = R$ (radius of earth)

$$\boxed{v_o = \sqrt{gR}}$$

2. Time period of the satellite orbit:

The orbital velocity of the satellite is $v_o = \sqrt{\frac{GM}{r}}$ ----- (1)

If ω is the angular velocity of the satellite, then $v_o = r\omega$ ----- (2)

Comparing (1) and (2) $r\omega = \sqrt{\frac{GM}{r}}$

or $\omega = \sqrt{\frac{GM}{r^3}}$ by definition $\omega = \frac{2\pi}{T}$ where T is the time period of the satellite.

Thus $\frac{2\pi}{T} = \sqrt{\frac{GM}{r^3}}$ or $T = 2\pi\sqrt{\frac{r^3}{GM}}$

If h is the height of the satellite from the surface of earth and $R \rightarrow$ radius of earth, then

$$r = R + h \quad \text{Thus} \quad T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

As $GM = gR^2$ $T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$

If the satellite is very near surface of earth $h \ll R$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{R}{g}}}$$

Escape velocity: It is defined as the minimum velocity with which an object has to be projected from the surface of the planet so that it escapes the planet's gravitational force of attraction.

If m is mass of an object projected from surface of earth, and $M \rightarrow$ mass of earth, $R \rightarrow$ radius of earth and v is the velocity of projection, then total energy of the object at the surface of the earth (just after projection) is

$$TE = (KE + PE)_{r=R} = \frac{1}{2}mv^2 - \frac{GMm}{R} \text{ ----- (1)}$$

where $KE = \frac{1}{2}mv^2$ and $PE = \frac{-GMm}{R}$

If v_e is the velocity of the object just enough to take the object to infinity ($v = v_e$), as both kinetic and potential energies are zero at infinity, then

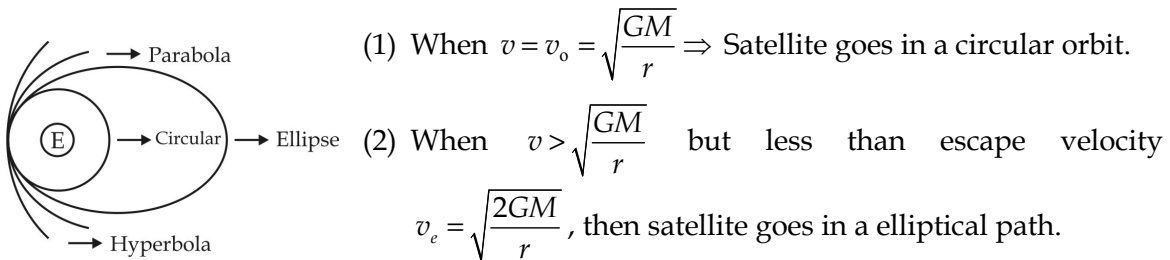
$$TE = (KE + PE)_{r=\infty} = 0 \text{ ----- (2)}$$

From the law of conservation of energy (1) = (2)

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0 \quad \text{or} \quad \frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

or $v_e^2 = \frac{2GM}{R}$ or $v_e = \sqrt{\frac{2GM}{R}}$ In general $v_e = \sqrt{\frac{2GM}{r}}$

The condition for launching artificial satellites:



(3) When $v = v_e = \sqrt{\frac{2GM}{r}}$, the satellite takes a parabolic path and escapes to infinity.

(4) When $v > \sqrt{\frac{2GM}{r}}$, the satellite takes a hyperbolic path and escapes to infinity.

Geostationary satellite: A satellite that appears to be fixed at a position above a certain distance from earth having same period of rotation of earth (24 hr) is called geo stationary satellite.

The satellite rotates in the plane of the equator with the same rotation as that of earth and thus appears to be stationary to an observer on the earth.

The angular velocity of the satellite is $\omega = \frac{2\pi}{T}$.

where $T \rightarrow$ time period of geo stationary satellite.

As $T = 24 \text{ hrs} = 86,400 \text{ sec}$, $\omega = \frac{2\pi}{86,400} = 7.3 \times 10^{-5} \text{ rad s}^{-1}$.

The orbital velocity $v_o = \sqrt{\frac{GM}{r}}$.

As $v_o = r\omega = \sqrt{\frac{GM}{r}} \Rightarrow \omega = \sqrt{\frac{GM}{r^3}}$ or $\omega^2 = \frac{GM}{r^3}$

or $r^3 = \frac{GM}{\omega^2}$ or $r = \left[\frac{GM}{\omega^2}\right]^{\frac{1}{3}}$

$G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$, $M = 5.98 \times 10^{24} \text{ kg}$ and $\omega = 7.3 \times 10^{-5} \text{ rad s}^{-1}$

$$r = \left[\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(7.3 \times 10^{-5})^2} \right]^{\frac{1}{3}}$$

$$r = 42 \times 10^6 \text{ m}$$

The height of the geo stationary satellite above the surface of the earth is given by

$$h = r - R \quad [\because r = R + h] \quad R \rightarrow \text{radius of earth} \quad R = 6.37 \times 10^6 \text{ m}$$

$$h = [42 \times 10^6 - 6.37 \times 10^6]$$

$$h = 35.63 \times 10^6 \text{ m} \quad \text{or} \quad \boxed{h = 35,630 \text{ km}}$$