Concept of Rest and Motion

A body is said to be at rest if its position does not change with time with respect to an observer (or a reference point). For example, the passengers sitting in a moving bus are said to be at rest with respect to the driver of the same bus, because their positions do not change with respect to the driver.

A body is said to be in motion if its position changes with time with respect to an observer (or a reference point). The passengers sitting in a moving bus are said to be in motion with respect to an observer standing outside the bus. Similarly, the blades of a rotating fan, the hands of a working wall clock, a spinning top and satellites are all in motion with respect to a fixed axis.

Rest and motion are relative terms. A body may seem to be at rest with respect to one object, but may appear to be in motion with respect to another object. If you consider a passenger in a moving train, he is at rest with respect to his co-passengers, but is in motion with respect to an observer standing on the ground.

Frames of reference : A system of co-ordinate axis which defines the position of a particle in two or three dimensional space is called a frame of reference. The simplest frame of reference is the Cartesian system of co-ordinates, in which the position of the particle is specified by its three co-ordinates x,y,z along the three perpendicular axes. In general, it is a framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system.

Inertial and non inertial frames of reference

Inertia frames of reference are those reference frames in which Newton's laws are valid. They are non-accelerating frames or constant velocity frames. Such a constant velocity frame of reference is called an *inertial* frame because the law of inertia holds in it. In an inertial frame of reference no fictitious forces arise.

Examples : A space shuttle moving with constant velocity relative to the earth, a rocket moving with constant velocity relative to the earth, any reference frame that is not accelerating, a frame attached to a particle on which there are no forces, any reference frame that is at rest, a reference frame attached to the center of the universe, a reference frame attached to Earth.

Non-inertial frame of reference are those reference frames in which Newton's laws are not valid. They are accelerating frames. Such an accelerating frame of reference is called a non-inertial frame because the law of inertia does not hold in it. That is, its velocity is *not* constant. So, it is either changing its speed by speeding up or slowing down, or it is changing its direction by traveling in a curved path, or it is both changing its speed and changing its direction. In a non-inertial frame of reference fictitious forces arise.
Examples : Elevator, Rotating frames, any accelerating frames, etc,…..

Explanation : If we are in a car when the brakes are abruptly applied, then we will feel pushed toward the front of the car. However, there is really no force pushing us forward. The car, since it is slowing down, is an accelerating, or non-inertial frame of reference. The law of inertia no longer holds in this non-inertial frame to judge our motion.
If this situation is viewed relative to the ground, which is at rest, it becomes clear that no force is pushing us forward when the brakes are applied. The ground is stationary and, therefore, is an inertial frame. Relative to this inertial frame, when the brakes are applied, we continue with our forward motion, just like we should according to Newton's first law of motion. The situation is this: the car is stopping but we are not. From our point of view in the car it seems like we have spontaneously been pushed forward. Actually, there is no force acting on us. The imagined force acting on us called **fictitious or pseudo force** or the frame dependent force.
A similar fictitious force can be noticed by a person in a car when it speeds up. Pseudo force is an imaginary force which is experienced only in a non-inertial frame of reference. The conclusion is that, in inertial frame of reference, zero force corresponds to zero acceleration. In non-inertial force, zero force doesn't correspond to zero acceleration.

Galilean relativity

Any two observers moving at constant speed and direction with respect to one another will obtain the same results for all mechanical experiments.
Explanation : Consider an aeroplane moving with a constant velocity. If a passenger in the aeroplane throws a ball straight up in the air, he observes the ball moving in a vertical

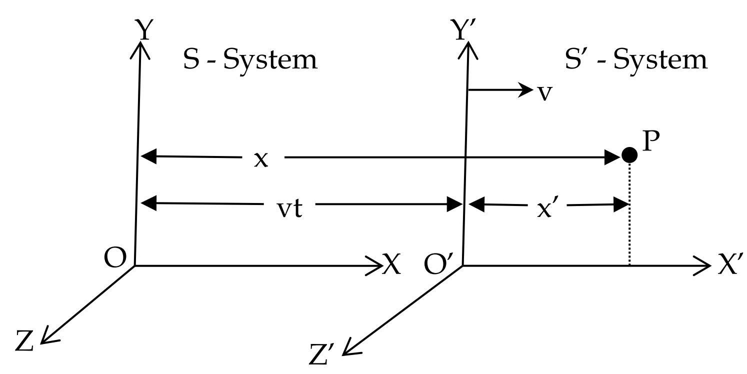
path. The motion of the ball is precisely the same as it would be if the ball were thrown while at rest on the Earth. The law of gravity and the equations of motion under constant acceleration are obeyed whether the aeroplane is at rest or in uniform motion.

Now consider the same experiment when viewed by another observer at rest on Earth. This stationary observer views the path of the ball in the plane to be a parabola and the ball has a velocity to the right equal to the velocity of the plane. Although the two observers disagree on the shape of the ball's path, both agree that the motion of the ball obeys the law of gravity and Newton's laws of motion and even agree on how long the ball is in the air.

Conclusion: There is no preferred frame of reference of describing the laws of mechanics. Thus the fundamental physical laws and principles are identical in all inertial frames of reference. This is also the Newtonian relativity.

Galilean transformations are a set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other. They are adequate to describe phenomena at speeds much smaller than the speed of light. Galilean transformations formally express the ideas that space and time are absolute; that length, time, and mass are independent of the relative motion of the observer; and that the speed of light depends upon the relative motion of the observer.

Galilean transformation equations



Let S and S′ be two inertial frames of reference. Let S be at rest and S′ be moving with a uniform velocity v with respect to S along the positive X - direction. Also $v \ll c$ where c is the speed of light.

Let the origins of the two frames coincide at time t = 0. Suppose an event occurs at a point P. The observer O in frame S determines the position of the event as coordinates x, y, z . The observer O′ in frame S′ determines the position of the event as coordinates x', y', z' . Let the time proceed at the same rate in both the frames. At a later instant of time, the distance between the two frames is vt. From the above diagram, $OP = OP' + OO'$.

As $P = x, O'P = x' and\ OO' = vt$, the relation becomes

$x = x' + vt$ or $x' = x - vt$

As there is no relative motion between S and S′ along Y and Z directions,

$y = y'$ and $= z'$.

Thus the transformation equations given from S to S′ are

$\boldsymbol{x' = x - vt}$, $\boldsymbol{y' = y}$, $\boldsymbol{z' = z}$ and $\boldsymbol{t' = t}$ ……(1)

These are called Galilean positon transformation equations.

The inverse transformation from S′ to S are given by

$\boldsymbol{x = x' + vt}$, $\boldsymbol{y' = y}$, $\boldsymbol{z' = z}$ and $\boldsymbol{t' = t}$

Differentiating equation (1) with respect to time t, we get

$\frac{dx'}{dt} = \frac{dx}{dt} - v$, $\frac{dy'}{dt} = \frac{dy}{dt}$ and $\frac{dz'}{dt} = \frac{dz}{dt}$

Let $\frac{dx'}{dt} = u'$ be the velocity of the particle at P as measured by S′, $\frac{dx}{dt} = u$ be the velocity of the particle at P as measured by S, then the equations become

$\boldsymbol{u' = u - v}$, $\boldsymbol{\frac{dy'}{dt} = \frac{dy}{dt}}$ and $\boldsymbol{\frac{dz'}{dt} = \frac{dz}{dt}}$ ….(2) These equations are called non relativistic ($v \ll c$) Galilean velocity transformation equations.

The equation $\boldsymbol{u' = u - v}$ can be expressed as $\boldsymbol{u = u' + v}$ i.e., the velocity of a particle as measured by a stationary frame is equal to the sum of the velocity of the same particle as measured from the moving frame and velocity of S′ frame with respect to S frame. If $u = 0$, i.e. the particle is at respect to S frame, then $u' = -v$. Thus the particle appears to move with uniform velocity in S′ frame. Thus the Newton's first law is obeyed in both the frames.

Again differentiating equation (2) $\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}$, $\frac{d^2y'}{dt^2} = \frac{d^2y}{dt^2}$, $\frac{d^2z'}{dt^2} = \frac{d^2z}{dt^2}$. ($\boldsymbol{v}$ is a constant).

As $\frac{d^2x'}{dt^2} = a'$ is the acceleration of the particle as measured by S′ frame and $\frac{d^2x}{dt^2} = a$ is the acceleration in S frame.

Thus $\boldsymbol{a' = a}$ ….(3) Thus the acceleration of the particle as measured by the two frames are the same or **acceleration is invariant.**

Equation (3) is the Galilean acceleration transformation equation.

Multiplying the above equation by m, the mass of the particle, we get $\boldsymbol{ma'} = \boldsymbol{ma}$ or $\boldsymbol{F'} = \boldsymbol{F}$. Thus the Newton's laws are valid in S′ frame also and thus a inertial frame. Thus the laws of mechanics are same in all inertial frames of reference which is the principle of Galilean relativity.

The concept of Galilean relativity does not apply to experiments in electricity, magnetism, optics and other areas. For eg. if we assume that the laws of electricity and magnetism are the same in all inertial frames, a paradox concerning the speed of light arises. This can be understood by recalling that according to electromagnetic theory, the speed of light always has the fixed value of 2.99792458 x 10^8 ms^{-1} in free space. But this is in direct contradiction to common sense.

Suppose a light pulse is sent out by an observer S′ in a car moving with velocity v. The light pulse has a velocity c relative to observer S′. According to Galilean relativity, the velocity of the pulse relative to stationary observer S outside the car should be $c + v$. But it is wrong as the velocity of the pulse will still be c.

To resolve the paradox, a preferred reference frame must exist in which the speed of light has the value c, but in any other reference frames the speed of light must have a value of greater or less than c.

Electromagnetic theory predicted that electromagnetic waves must propagate through free space with a speed equal to the speed of light. However, the theory does not require the presence of a medium for wave propagation. Hence, physicists of the 19th century, proposed that electromagnetic waves also required a medium in order to propagate. This medium was called ether.

Properties of ether: Massless but rigid medium with no effect on the motion of other planets and are present everywhere even in empty space.

The laws of electricity and magnetism would take on their simplest forms in a special frame of reference at rest with respect to the ether. This frame was called the absolute frame. The laws of electricity and magnetism would be valid in this absolute frame, but they would have to be modified in any reference frame moving with respect to the absolute frame.

Michelson – Morley experiment is designed to determine the velocity of Earth relative to the hypothetical ether. When Earth moves through the ether, to an experimenter on Earth, there was an "ether wind" blowing through his lab. When the apparatus was rotated, the fringe pattern is supposed to shift slightly but measurably. However, no fringe shift of the magnitude required was observed. This result contradicted the ether hypothesis and showed that it was impossible to measure the absolute velocity of Earth with respect to the ether frame. There is no absolute fixed frame of reference.

Einstein's Special theory of Relativity

Einstein proposed his famous theory of special relativity in the year 1905 to overcome these difficulties. The part of relativistic mechanics which is related to uniform motion is called 'Special Theory of Relativity'. Einstein's special theory of relativity is based on the following fundamental postulates,

1. **Principle of Relativity - The laws of physics are the same in all inertial frames of reference.** There is no preferred or absolute inertial frame of reference i.e. all inertial frames are equivalent for the description of all physical laws such as Newton's laws as well as Maxwell's electromagnetic equations.
2. **Principle of constancy of speed of light - The speed of light in free space is constant.** i.e. the speed of light is the same for all observers in uniform translational relative motion and is independent of the motion of the observer and the source. **Speed of light is invariant**.

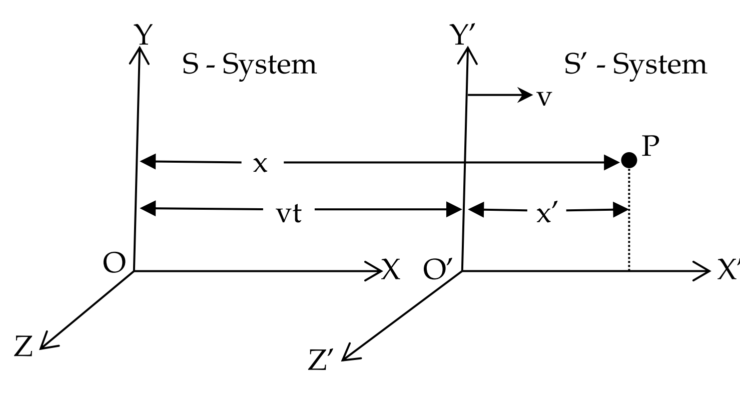
The result of Einstein's theory was to introduce new coordinate transformations, called Lorentz transformations, between inertial frames of reference. At slow speeds, these transformations were essentially identical to the classical model, but at high speeds, near the speed of light, they produced radically different results.

The first postulate is a generalization of the principle of Galilean relativity, which refers only to laws of mechanics. Einstein's principle of relativity means that any kind of experiment – mechanical, thermal, optical, or electrical – performed in a laboratory at rest must give the same result when performed in a laboratory moving at a constant speed past the first one. Hence no preferred inertial reference frame exists.

The second postulate says velocity of light is the same in all the inertial frames which was not the same in the Galilean transformations. This is the only postulate that differentiates the classical theory and the Einstein's theory of relativity.

Lorentz Transformation equations

Lorentz discovered new transformation equations which are consistent with the new concept of invariance of velocity of light in free space.



Let S and S′ be two inertial frames of reference. Let S be at rest and S′ be moving with a uniform velocity

v with respect to S along the positive X - direction. A light pulse produced at t = 0 will spread as a growing sphere. The radius of this sphere increases with speed c. After a time t, the observer will note that the light has reached a point $P(x, y, z)$ and for the observer it will reach the point $P(x', y', z')$ as shown. The transformation equation relating x and x′ can be written as

$x' = k(x - vt)$. Also $t' = k(t - bx)$, $y' = y$ and $z' = z$ where k and b are constants. They are given by $k = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $b = \frac{v}{c^2}$.

Substituting the constants in the transformation equations, we get

$$\boldsymbol{x' = \frac{(x-vt)}{\sqrt{1-\frac{v^2}{c^2}}}\ ,\quad y' = y\ ,\quad z' = z} \quad \text{and} \quad \boldsymbol{t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}} \qquad \text{........(1)}$$

These are called Lorentz transformation equations.

The inverse Lorentz transformation equations are

$$\boldsymbol{x = \frac{(x'+vt')}{\sqrt{1-(v^2/c^2)}}\ ,\quad y = y'\ ,\quad z = z'} \quad \textbf{and} \quad \boldsymbol{t = \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-(v^2/c^2)}}} \qquad \text{........(2)}$$

Note : To show that for speeds $v \ll c$, the Lorentz transformation reduces to Galilean transformation. When $v \ll c$, $\frac{v}{c} \to 0$. Thus $\frac{1}{\sqrt{1-(v^2/c^2)}} = 1$ The Lorentz equations reduces to $\boldsymbol{x' = x - vt}$, $\boldsymbol{y' = y}$, $\boldsymbol{z' = z}$ and $\boldsymbol{t' = t}$ which are the Galilean transformation equations.

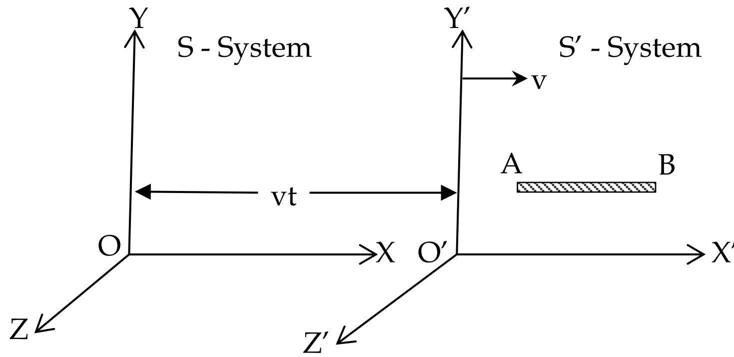
Effects of Special Relativity

Special relativity yields several consequences from applying Lorentz transformations at high velocities (near the speed of light). Among them are:

Time dilation (including the popular "twin paradox"), Length contraction, Velocity transformation, Relativistic velocity addition, Relativistic doppler effect, Simultaneity & clock synchronization, Relativistic momentum, Relativistic kinetic energy, Relativistic mass and Relativistic total energy

Length contraction

Let S and S′ be two inertial frames of reference. Let S be at rest and S′ be moving with a uniform velocity v with respect to S along the positive X - direction. Consider a rod AB of length L remaining at rest relative to the frame S′.



Let x_1' and x_2' be the coordinates (distance from the origin O′) of the ends of the rod in the frame S′. As the rod is at rest relative to S′, its length is called the proper length given by $L_0 = x_2' - x_1'$ ….(1)
Similarly the coordinates of the ends of the rod in the S frame x_1 and x_2 . Then the length of the rod as measured relative to S frame is $L = x_2 - x_1$.
From the Lorentz transformation equation,

$$x_1' = \frac{(x_1 - vt)}{\sqrt{1-(v^2/c^2)}} \text{ …(2) and } x_2' = \frac{(x_2 - vt)}{\sqrt{1-(v^2/c^2)}} \text{ …(3)}$$

Subtracting (2) from (3), we get $x_2' - x_1' = \frac{(x_2 - vt)}{\sqrt{1-(v^2/c^2)}} - \frac{(x_1 - vt)}{\sqrt{1-(v^2/c^2)}}$

or $x_2' - x_1' = \frac{x_2 - x_1}{1-(v^2/c^2)}$. Thus $L_0 = \frac{L}{1-(v^2/c^2)}$, or $\boldsymbol{L = L_0\sqrt{1-(v^2/c^2)}}$ …(4)

From equation (4) it is observed that $L < L_0$. Thus for an observer in S the length of the rod appears to be contracted or reduced by a factor of $\sqrt{1-(v^2/c^2)}$. This shortening or contraction of length of an object along its direction of motion is known as Lorentz – Fitzgerald contraction. The contraction becomes appreciable when $v \approx c$.

Time dilation

Consider a gun placed at a point in a frame S′ which is moving with uniform velocity v with respect to a frame S at rest. Let a clock in the moving frame S′ measure t_1' and t_2' as the times at which two shots are fired from the gun in frame S′. As this clock is at rest with respect to the observer in frame S′, the time interval between the two explosions is called the proper time interval $t_0 = t_2' - t_1'$. Since the gun is fixed in S′, it has a velocity v with respect to S in the positive X- direction. Let $t = t_2 - t_1$ represent the time interval between the two shots as measured by an observer in S.
From the inverse Lorentz transformation equation, we have

$$t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-(v^2/c^2)}} \text{ ….(1) and } t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-(v^2/c^2)}} \text{ ….(2)}$$

Subtracting (1) from (2), we get $t_2 - t_1 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1-(v^2/c^2)}} - \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1-(v^2/c^2)}}$

Thus $t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}}$ or $\boldsymbol{t = \frac{t_0}{\sqrt{1-(v^2/c^2)}}}$ …….(3)

Equation (3) indicates that $t > t_0$. Thus the time interval between two events occurring at a given point in the moving frame S′ appears to be longer for the observer in the stationary frame S. i.e. a stationary clock (frame S) measures a longer time interval between events occurring in a moving frame of reference than does the clock (frame S′) in the moving frame. This effect is called time dilation.

The twin paradox

Consider twins with one of them going for a long journey to space at a high speed in a rocket and the other remaining on the earth. The clock in the moving rocket appears to go slow for the observer on the earth (in accordance with $t = \frac{t_0}{\sqrt{1-(v^2/c^2)}}$). Therefore, when he returns back to the earth, he will find himself younger than the twin who stayed back on the earth.

Illustration of length contraction and time dilation – Meson decay

When the cosmic ray particles arriving at the earth from space, strike the atmosphere, unstable particles called μ - mesons are created. Their mean life time is 2μs. ($1\ \mu = 10^{-6}$) i.e. they decay to electrons in 2μs after coming into existence. These mesons reach the sea level in profusion. They have a speed of $2.994 \times 10^8 ms^{-1}$ which is 0.998c.

In this lifetime mesons travel a distance of $y = vt_0 = 2.994 \times 10^8 \times 2 \times 10^{-6} = 600m$. But mesons are created at heights much larger than this distance. This paradox is resolved from the special theory of relativity.

1. Consider the frame of reference fixed to the meson. The meson sees a shortened distance by a factor of $y/y_0 = \sqrt{1-(v^2/c^2)}$ due to length contraction. i.e. when we on the earth measure the altitude at which the meson is produced as y_0, but meson sees it as reduced altitude y (y is the maximum distance meson can travel before it decays). Thus with respect to earth $y_0 = \frac{y}{\sqrt{1-(v^2/c^2)}} = \frac{600}{\sqrt{1-(0.998c)^2/c^2}} = 9500\ m$. Thus meson can reach the earth despite its short life time.

2. Now let this problem be observed with respect to frame of reference of the earth. From the ground it is found that the meson is produced at an altitude y_0. But the lifetime of the meson with respect to the earth is extended due to time dilation to a value t given by $t = \frac{t_0}{\sqrt{1-(v^2/c^2)}} = \frac{2 \times 10^{-6}}{\sqrt{1- [0.998c]^2/c^2}} = 31.7 \times 10^{-6} s$. In this time the distance travelled by the meson as seen from the earth is $y_0 = vt = 2.994 \times 10^8 \times 31.7 \times 10^{-6} = 9500\ m$. This is the same as obtained earlier. This indicates, the two views are the same.

Relativity of simultaneity

Consider two events, i.e. two explosions occurring at two different locations x_1 and x_2 at the same time t_0 as measured by an observer O in frame S which is at rest. Consider another observer O′ in the frame S′ moving with relative constant velocity v with respect to the frame S along the positive X - direction. To the observer O′ in S′ the explosion at x_1 occurs at the time $t_1' = \frac{t_0 - \frac{vx_1}{c^2}}{\sqrt{1-(v^2/c^2)}}$ and the explosion at x_2 occurs at time $t_2' = \frac{t_0 - \frac{vx_2}{c^2}}{\sqrt{1-(v^2/c^2)}}$. The time interval between the two events as observed by O′ in S′ is $t_2' - t_1' = \frac{\frac{v}{c^2}(x_1 - x_2)}{\sqrt{1-(v^2/c^2)}}$.

This is not equal to zero. This indicates the two events at x_1 and x_2 which are simultaneous for an observer O in S is not simultaneous for the observer O′ in S′. Thus the concept of simultaneity is only relative and not absolute.

Variation of mass with velocity

The relativistic formula for the mass m of a body moving with a velocity v is given by $= \frac{m_0}{\sqrt{1-(v^2/c^2)}}$. When the velocity $v \rightarrow c,$ i.e. the body travelling at the speed of light, the mass becomes infinite. Thus no material particle can travel at the speed of light or greater than the speed of light.

Mass Energy equivalence – Derivation of $E = mc^2$

Consider a body of mass m moving with velocity . Let F be the force acting on the body. From Newton's second law, force is equal to rate of change of momentum, given by $F =$

$\frac{d}{dt}(mv)$ ….(1) . According to the special theory of relativity, both mass and velocity are variables. Therefore

$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$ ….(2)

Let the force F displace the body by a distance, dx. Then, the increase in the kinetic energy (dE_k) of the body is equal to the amount of work done ($F\ dx$) .

Hence, $dE_k = F\ dx$ ….(3) Substituting for F from (2) in (3), we get

$dE_k = m\frac{dv}{dt}dx + v\frac{dm}{dt}\ dx = m\frac{dx}{dt}dv + v\frac{dx}{dt}\ dm$

As $v = \frac{dx}{dt}$, the above equation becomes $dE_k = mvdv + v^2\ dm$ ….(4)

According to the law of variation of mass with velocity $m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$

Squaring this equation and rearranging, $m^2 = \frac{{m_0}^2}{1-\frac{v^2}{c^2}} = \frac{{m_0}^2}{\frac{c^2-v^2}{c^2}} = \frac{{m_0}^2c^2}{c^2-v^2}$

Thus $m^2 = \frac{{m_0}^2c^2}{c^2-v^2}$ or $m^2(c^2 - v^2) = {m_0}^2c^2$ or $m^2c^2 - m^2\,v^2 = {m_0}^2c^2$

or $m^2c^2 = {m_0}^2c^2 + m^2\,v^2$

Differentiating this equation, $c^2 2m\,dm = 0 + m^2 2v\,dv + v^2 2m\,dm$

Dividing this equation by $2dm$, we get, $c^2dm = mv\,dv + v^2dm$ ….(5)

Comparing (4) and (5), we get $dE_k = c^2dm$ …..(6)

Thus the change in kinetic energy is directly proportional to the change in mass, dm.

When a body is at rest, its velocity is zero. Its KE is zero and the mass is $m = m_0$.

When the velocity of the body is v, its mass is m. Therefore integrating equation (6)

$\int_0^{E_k} dE_k = c^2\int_{m_0}^{m} dm$ Thus $E_k = c^2(m - m_0)$

or $\boldsymbol{E_k = mc^2 - m_0c^2}$ ….(7) This is the **relativistic equation for kinetic energy** of the particle.

When the body is at rest the internal energy stored in the body is m_0c^2 which is called the rest mass energy. Thus the total energy (E) of the body is the sum of the KE (E_k) and the rest mass energy (m_0c^2).

The total energy is $E = E_k + m_0c^2$. Substituting for E_k from (7) in this equation, we get

$E = mc^2 - m_0c^2 + m_0c^2$ or $\boldsymbol{E = mc^2}$

This is called the **Einstein's mass energy relation**. This relates the universal equivalence between mass and energy. i.e. mass and energy are interconvertible.

Examples : 1. Annihilation of matter and Pair production - Every known particle has an antiparticle. If they encounter one another, they will annihilate with the production of two gamma-rays. An electron and a positron combines and vanishes literally and results in gamma radiation. The quantum energies of the gamma rays is equal to the sum of the mass energies of the two particles (including their kinetic energies). It is also possible for a photon to give up its quantum energy to the formation of a particle-antiparticle pair in its interaction with matter.

2. The equation $E = mc^2$ is the basis of understanding the nuclear reactions such as nuclear fission and fusion.

Note : The formula for the kinetic energy reduces to classical formula for $v \ll c$.

$$E_k = mc^2 - m_0c^2 \qquad or \qquad E_k = (m - m_0)c^2 = \left(\frac{m_0}{\sqrt{1-(v^2/c^2)}} - m_0\right)c^2$$

or $E_k = m_0c^2 \left[\left(1 - (v^2/c^2)\right)^{-1/2} - 1\right]$

For $v \ll c$, $\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{v^2}{2c^2}$ ($(1+x)^n = 1 + nx + \cdots$. higher terms are neglected).

Thus $E_k = m_0c^2\left[1 + \frac{v^2}{2c^2} - 1\right] = m_0c^2 \times \frac{v^2}{2c^2} = \frac{1}{2}\, m_0v^2$.

Relationship between total energy, rest mass energy and the momentum

The total relativistic energy of a particle is $E = mc^2 = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}}$.

The momentum of the particle is $p = mv$ or $v = p/m$. Substituting for v in the above equation, we get $E = \frac{m_0c^2}{\sqrt{1-(p^2/m^2c^2)}} = \frac{m_0c^2}{\sqrt{1-(p^2c^2/m^2c^4)}} = \frac{m_0c^2}{\sqrt{1-\left(p^2c^2/E^2\right)}}$ $(\because E = mc^2)$

Thus $E = \frac{m_0c^2}{\sqrt{1-\left(p^2c^2/E^2\right)}}$ or $E^2 = \frac{m_0^2c^4}{1-(p^2c^2/E^2)}$ or $E^2 - p^2c^2 = m_0^2c^4$

Thus the relation between the total energy, rest energy and the momentum is given by $\boldsymbol{E^2 = m_0^2c^4 + p^2c^2}$.