5. FOURIER SERIES

5.1 Introduction

In various engineering problems it will be necessary to express a function in a series of sines and cosines which are periodic functions. Most of the single valued functions which are used in applied mathematics canbe expressed in the form.

$$\frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + \ldots\ldots$$

within a desired range of values of x. Such a series is called a **Fourier Series** in the name of the French mathematician Jacques Foureier (1768 - 1830)

5.2 Periodic Functions

Definition : If at equal intervals of the abscissa 'x' the value of each ordinate f(x) repeats itself then f(x) is called a **periodic function.** i.e., A function f(x) is said to be a **periodic function** if there exists a real number a such that f(x + a) = f(x) for all x. The number a is called the period of f(x).

$\therefore$ we have f(x) = f(x + a) = f(x + 2a) = f(x + 3a)

= …………………..= f(x + n a) = ………….

Ex : (i) sin x = sin (x + 2p) = sin (x +4p) = ……………

.......….= sin (x + 2n p) = ……………

Hence sin x is a periodic function of the period 2 p .

(ii) cos x = cos(x + 2p) = cos (x + 4p) = ………

……….. = cos (x + 2n p) = …………….

Hence cos x is a periodic function of the period 2 p .

We define the Fourier series in terms of these two periodic functions.

5.3 Fourier Series

Definition : A series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{np\,x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{np\,x}{l})$$

is called a **Fourier series** of f(x) with period $2l$ in the interval (c, c +2l) where l is any positive real number and a_0, a_n, b_n are given by the formulae called **Euler's Formulae** :

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos(\frac{np\,x}{l})dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin(\frac{np\,x}{l})dx$$

These coefficients a_0, a_n, b_n are known as **Fourier coefficients.**

In particular if $l = p$, the Fourier series of f(x) with period 2p in the interval (c, c+2p) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

and the Fourier coefficients are given by

$$a_0 = \frac{1}{p}\int_c^{c+2p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_c^{c+2p} f(x)\cos np\ dx$$

$$b_n = \frac{1}{p}\int_c^{c+2p} f(x)\sin np\ dx$$

We shall derive the Euler's formulae' for which the following definite integrals are required.

(i) $\int_{c}^{c+2l} dx = 2l$

(ii) $\int_{c}^{c+2l} \cos\frac{mp\,x}{l}dx = \int_{c}^{c+2l} \sin\frac{mp\,x}{l}dx = 0$

(iii) $\int_{c}^{c+2l} \cos\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx = 0$ for all integers m and n

(iv) $\int_{c}^{c+2l} \cos\frac{mp\,x}{l}\cos\frac{np\,x}{l}dx = \int_{c}^{c+2l} \sin\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx = 0$

(for all integers m and n such that $m \neq n$)

(v) $\int_{c}^{c+2l} \cos^2\frac{mp\,x}{l}dx = l = \int_{c}^{c+2l} \sin^2\frac{mp\,x}{l}dx$

5.4 Derivation of Euler's Formulae

We have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{np\,x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{np\,x}{l}$ . . .(1)

To find the coefficients a_0, a_n and b_n, we assume that the series (1) can be integrated term by term from x = c to x = c + 2l

To find a_0, integrate (1) w.r.t x from c to $c + 2l$.

$$\therefore \int_{c}^{c+2l} f(x) = \frac{a_0}{2}\int_{c}^{c+2l} 1dx + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2l} \cos\left(\frac{np\,x}{l}\right)dx + \sum_{n=1}^{\infty} b_n \int_{c}^{c+2l} \sin\left(\frac{np\,x}{l}\right)dx$$

$$= \frac{a_0}{2}(2l) + \sum_{n=1}^{\infty} a_n(0) + \sum_{n=1}^{\infty} b_n(0)$$

$= a_0(l)$ (using the definite integrals (ii) above)

$$\therefore a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)dx \quad . . . (a)$$

To find a_n, multiply both sides of (1) by $\cos\frac{mp\,x}{l}$ where m is a fixed positive integer and integrate w.r.t x from $x = c$ to $x = c + 2l$

$$\therefore \int_{c}^{c+2l} f(x)\cos\frac{mp\,x}{l}dx$$

$$= \frac{a_0}{2}\int_{c}^{c+2l} \cos\frac{mp\,x}{l}dx + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2l} \cos\frac{mp\,x}{l}\cos\frac{np\,x}{l}dx + \sum_{n=1}^{\infty} b_n \int_{c}^{c+2l} \cos\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx$$

$$= \frac{a_0}{2}(0) + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2l} \cos\frac{mp\,x}{l}\cos\frac{np\,x}{l}dx + \sum_{n=1}^{\infty} b_n(0)$$

[Using the definite integrals (ii) and (iii) above]

$$= \sum_{n=1}^{\infty} a_n \int_{c}^{c+2l} \cos\frac{mp\,x}{l}\cos\frac{np\,x}{l}dx \quad (m \neq n)$$

$$+ a_m \int_{c}^{c+2l} \cos^2\frac{mp\,x}{l}dx \quad (m = n)$$

$$= \sum_{n=1}^{\infty} a_n(0) + a_m(l)$$

[Using the definite integrals (iv) and (v) above]

$= a_m(l)$

$$\therefore a_m = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\frac{mp\,x}{l}dx$$

Changing m to n we get

$$a_m = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{np\,x}{l}dx \qquad \ldots(b)$$

To find b_n, multiply both sides of (1) by $\sin\frac{mp\,x}{l}$ where m is a fixed positive integer and integrate w.r.t x from $x = c$ *to* $x = c + 2l$

$$\therefore \int_c^{c+2l} f(x)\sin\frac{mp\,x}{l}dx$$

$$= \frac{a_0}{2}\int_c^{c+2l}\sin\frac{mp\,x}{l}dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2l}\sin\frac{mp\,x}{l}\cos\frac{np\,x}{l}dx$$

$$+\sum_{n=1}^{\infty} b_n \int_c^{c+2l}\sin\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx$$

$$= \frac{a_0}{2}(0) + \sum_{n=1}^{\infty} a_n(0) + \sum_{n=1}^{\infty} b_n \int_c^{c+2l}\sin\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx$$

[Using the definite integrals (ii) and (iii) above]

$$= \sum_{n=1}^{\infty} a_n \int_c^{c+2l}\sin\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx \quad (m \neq n)$$

$$+ b_m \int_c^{c+2l}\sin\frac{mp\,x}{l}\sin\frac{np\,x}{l}dx \quad (m = n)$$

$$= 0 + b_m \int_c^{c+2l}\sin^2\frac{mp\,x}{l}dx$$

[Using the definite integrals (iv) above]

$= b_m(l)$ [using the definite integral (v)]

$$\therefore b_m = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{mp\,x}{l}dx$$

Changing m to n we get

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{np\,x}{l}dx \qquad (b)$$

Thus the Euler's formulae (a), (b), (c) are proved.

Cor. 1 : In particular if $l = p$ and $c = 0$, we get the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where the Foureir coefficients are given by

$$a_0 = \frac{1}{p}\int_0^{2p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_0^{2p} f(x)\cos np\ dx$$

$$b_n = \frac{1}{p}\int_0^{2p} f(x)\sin np\ dx$$

Cor. 2: In the above formulae if $l = p$ and $c = -p$, we get the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx,$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos np\ dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin np\ dx$$

5.5 Conditions for a Fourier series expansion

It should not be mistaken that every function can be expanded as a Fourier series. In the above formulae we have only shown that if f(x) is expressed as a Fourier series, then the Fourier coefficients are given by Euler's formula. It is very cumbersome to discuss whether a function can be expressed as a Fourier series and to discuss the convergence of this series. However the following condition called Dirichlet's condition cover all problems.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{np\,x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{np\,x}{l}$$

provided

- (i) f(x) is bounded
- (ii) f(x) is periodic, single – valued and finite
- (iii) f(x) has a finite number of discontinuities in any one period.
- (iv) f(x) has at the most a finite number of maxima and minima.

These conditions are called **Dirichlets** conditions. In fact expressing a function f(x) as a Fourier series depends on the evaluation on the definite integrals

$$\frac{1}{l}\int f(x)\cos\frac{np\,x}{l}\,dx \text{ and } \frac{1}{p}\int f(x)\sin\frac{np\,x}{l}\,dx$$

within the limits c to c + 2*l*, 0 to 2*p* or -*p* to *p* according as f(x) is defined for all x in (*c*, *c* + 2*l*) (0, 2*p*) or (-*p* ,*p*)

5.6 Interval with 0 as mid point

If c = -*l* then the interval (*c*, *c* + 2*l*) becomes (-*l*, *l*) and further if c = -*p* , the interval becomes (-*p* ,*p*). These intervals have 0 as the mid point. For functions defined in such intervals, we consider the effect of changing x to –x and classify them as even and odd functions.

5.7 Even and odd functions

A function f(x) is said to be even if f(-x) = f(x) $\forall x$ in the given interval (*c*, *c* + 2*l*) and a function f(x) is said to be odd if f(-x) = -f(x) $\forall x$ in the given interval (*c*, *c* + 2*l*)

5.7.1 Tests for even and odd mature of a function

If f(x) is defined by one single expression, f(-x) = f(x) implies f (x) is even and f(-x) = -f(x) implies f(x) is odd. If f(x) is defined by two or more expressions on parts of the given interval with 0 as the mid point, f(-x) from the function as defined on one side of 0 = f(x) from the corresponding function as defined on the other side, implies f(x) is even.

f(-x) from the function as defined on one side of 0 = -f(x) from the corresponding function as defined on the other side, implies f(x) is odd.

Examples :

(1) f(x) = x^2 + 1 in (-1, 1)

f(-x) = $(-x)^2 + 1 = x^2 + 1$ = f(x)

$\therefore$ f(x) is even.

(2) f(x) = x^3 in (-1, 1)

f(-x) = $(-x^3) = -x^3 = -$ f(x)

$\therefore$ f(x) is odd.

(3) $f(x) = \begin{cases} x+1 & in\ (-p,0) \\ x-1 & in\ (0,p) \end{cases}$

$$f(-x) \text{in } (0,p) = -x-1 = -(x+1) = -f(x) \text{ in } (-p,0)$$

$\therefore$ f(-x) = -f(x)

$\therefore$ f(x) is odd

5.7.2 Fourier coefficients when f(x) is even and odd

From definite integrals, we have

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(x) \text{ is even.}$$

$$\text{and } \int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd.}$$

(a) If f(x) is even in (-*l*, *l*) i.e., iff f(-x) = f(x), then

f(x) $\cos \frac{np\,x}{l}$ is also even.

$\therefore$ f(-x) $\cos \frac{np\,(-x)}{l}$ = f(x) $\cos \frac{np\,x}{l}$. Since cos(-*q*)= cos *q*

and f(x) $\sin \frac{np\,x}{l}$ is odd.

$\because$ f(x) $\sin \frac{np\,(-x)}{l} = -f(x)\sin \frac{np\,x}{l}$ since sin(-*q*) = -sin *q*

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_{0}^{l} f(x)dx \text{ (by above definite integral)}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{np\,x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{np\,x}{l}dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{np\,x}{l}dx = 0$$

$$\therefore f(x) = \frac{2}{l}\int_{0}^{l} f(x)dx + \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{np\,x}{l}dx$$

In this case if the interval is (-π, π) we get

$$a_0 = \frac{2}{p}\int_{0}^{p} f(x)dx$$

$$a_n = \frac{2}{p}\int_{0}^{p} f(x)\cos nx\,dx$$

$$b_n = 0$$

(b) If f(x) is odd in (-*l*, *l*) i.e., if f(-x) = -f(x) then

f(x) $\cos \frac{np\,x}{l}$ is also odd in (-*l*, *l*)

$$\because f(-x)\cos\frac{np\,(-x)}{l} = -f(x)\cos\frac{np\,x}{l}$$

and $f(x)\sin\frac{np\,x}{l}$ is even in (-*l*, *l*)

$$f(-x)\sin\frac{np\,(-x)}{l} = f(x)\sin\frac{np\,x}{l}$$

$$\therefore a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{np\,x}{l}dx = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{np\,x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{np\,x}{l}dx$$

If the interval is (-π, π) then a_0 = 0, a_n = 0

$$b_n = \frac{2}{p}\int_{0}^{x} f(x)\sin nx\,dx$$

5.7.3 Intervals with 0 as an end point

Intervals like (0, 2*l*) and (0, 2π) with 0 as end point have special features.

We know that $\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if $f(2a-x) = f(x)$

and $= 0$ if $f(2a-x) = -f(x)$

If f(2*l* - x) = f(x)

Then $a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{np\,x}{l}dx$$

$$b_n = 0$$

Similarly if $l = \pi$, i.e., if the interval is $(0, 2\pi)$ we get

$$a_0 = \frac{2}{p}\int_0^p f(x)dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos nx\,dx$$

$$b_n = 0$$

If $f(2l - x) = -f(x)$ then

$a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{np\,x}{l}\,dx$

Similarly If $f(2\pi - x) = -f(x)$ then

$a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{p}\int_0^p f(x)\sin nx\,dx$

WORKED EXAMPLES

1) Find the Fourier coefficient a_0 for $f(x) = x \sin x$ in $(0, 2\pi)$

(May 2003)

$$a_0 = \frac{1}{p}\int_0^{2p} x\sin x dx$$

$$= \frac{1}{p}[x(-\cos x) + \int \cos x.1dx]$$

$$= \frac{1}{p}[-x\cos x + \sin x]_0^{2p} = -2$$

2) Find the coefficient a_0 for $f(x) = x-1$ in $(-\pi,\pi)$ (A 1999)

$$a_0 = \frac{1}{p}\int_{-p}^{p}(x-1)dx$$

$$= \frac{1}{p}\left[\frac{x^2}{2} - x\right]_{-p}^{p}$$

$$= \frac{1}{p}\left[\frac{p^2}{2} - p\right] - \frac{1}{p}\left[\frac{p^2}{2} + p\right]$$

$= -2$

3) If $\begin{cases} 0 & for \quad -2 < x < 0 \\ 1 & for \quad 0 < x < 2 \end{cases}$

find the Fourier coefficient a_n in the fourier series.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{np\,x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0dx + \frac{1}{2}\int_0^2 1.\cos(\frac{np\,x}{2})dx$$

$$= \frac{1}{2}\left.\frac{\sin(\frac{np\,x}{2})}{\frac{np}{2}}\right]_0^2 = \frac{1}{np}[\sin(np) - 0] = 0$$

4) Obtain the Fourier series for $f(x) = x-1$ in the interval $(-\pi, \pi)$.

(A 1999)

Solution :

$$a_0 = \frac{1}{p}\int_{-x}^{x} f(x)dx = \frac{1}{p}\int_{-x}^{x}(x-1)dx$$

$$= \frac{1}{p}\int_{-x}^{x} xdx = \frac{1}{p}\int_{-x}^{x} dx$$

$$= 0 - \frac{1}{p}(x)_{-p}^{p} = \frac{1}{p}[2p] = -2$$

$$\therefore a_0 = -2$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos nx dx$$

$$= \frac{1}{p}\int_{-p}^{p} (x-1)\cos nx dx$$

$$= \frac{1}{p}\left[\int_{-p}^{p} x\cos nx - \int_{-p}^{p} \cos nx dx\right]$$

$$= \frac{1}{p}\left[0 - 2\int_{-p}^{p} \cos nx dx\right]$$

$$= \frac{1}{p}\left[\frac{-2\sin x}{n}\right]_0^p$$

$$= -\frac{2}{p}\left[\frac{\sin nx}{n} - \frac{\sin 0}{n}\right] = 0$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin nx dx$$

$$= \frac{1}{p}\int_{-p}^{p} (x-1)\sin nx dx$$

$$= \frac{1}{p}\left[\int_{-p}^{p} x\sin nx - \int_{-p}^{p} \sin nx dx\right]$$

$$= \frac{1}{p}\left[2\int_{0}^{p} x\sin nx dx - 0\right]$$

$$= \frac{2}{p}\left[x\left(\frac{-\cos nx}{n}\right) - \int -\left(\frac{\cos nx}{n}\right)dx\right]_0^p$$

$$= \frac{2}{p}\left[-\frac{x\cos nx}{n} - + \frac{\sin nx}{n^2}\right]_0^p$$

$$= \frac{2}{p}\left[\left(-\frac{p\cos np}{n} + 0\right) - (-0+0)\right]$$

$$= -\frac{2\cos np}{n} = -\frac{2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n}$$

$\therefore$ Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= -\frac{2}{2} + \sum_{n=1}^{\infty} 0 + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

$\therefore f(x) = -1 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$ is the required Fourier series

5) Expand f(x) = x^2 as a Fourier series in the interval $(-\pi, \pi)$ and hence show that (*i*) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \ldots = \frac{p^2}{12}$

(*ii*) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{p^2}{6}$ (A 1999)

Solution :

f(x) = x^2

$\therefore f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is even in $(-p, p)$

$\therefore b_n = 0$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{1}{p}\int_{-p}^{p} x^2 dx$$

$$= \frac{2}{p}\int_{0}^{p} x^2 dx = \frac{2}{p}\left[\frac{x^3}{3}\right]_0^p = \frac{2}{p}\frac{p^3}{3} = \frac{2p^2}{3}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos nx dx$$

$$= \frac{1}{p}\int_{-p}^{p} x^2 \cos nx dx = \frac{2}{p}\int_{0}^{p} x^2 \cos nx dx \quad \because x^2 \cos nx \text{ is even}$$

$$= \frac{2}{p}\left[x^2 \frac{\sin nx}{n} - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^p$$

$$= \frac{2}{p}\left[\left(p^2 \frac{\sin np}{n} + 2p\frac{\cos np}{n^2} - 2\frac{-\sin np}{n^3}\right) - 0\right]$$

$$= \frac{2}{p}\left[0 + 2p\frac{(-1)^n}{n^2} - 0\right]$$

$$i.e,\ a_n = \frac{4(-1)^n}{n^2}$$

$\therefore$ Foureir series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{p^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + 0$$

$$\therefore f(x) = \frac{p^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx \text{ is the required Fourier series.}$$

(i) To prove $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \ldots = \frac{p^2}{12}$

Put x = 0 in the above Fourier series

$$\therefore f(0) = \frac{x^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos 0$$

$$\therefore 0 = \frac{x^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \because f(0) = 0^2 = 0$$

$$i.e,\ 0 = \frac{x^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - + \ldots\right]$$

$$i.e,\ 0 = \frac{x^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - + \ldots\right]$$

$$\therefore 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - \ldots\right] = \frac{p^2}{3}$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{p^2}{12}$$

(ii) To prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{p^2}{6}$

Put x = p in the Fourier series of f(x)

$$f(p) = \frac{p^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos np$$

$$i.e,\ p^2 = \frac{p^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}(-1)^n$$

$$= \frac{p^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{2n}$$

$$= \frac{p^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \because (-1)^{2n} = 1$$

$$\therefore 4\sum_{n=1}^{\infty} \frac{1}{n^2} = p^2 - \frac{p^2}{3}$$

$$\text{i.e.,}\ 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] = 2\frac{p^2}{3}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{p^2}{6}$$

6) Obtain the Fourier series for f(x) = e^x in $(-p, p)$

Solution :

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx = \frac{1}{p}\int_{-p}^{p} e^x dx$$

$$= \frac{1}{p}\left[e^x\right]_{-p}^{p}$$

$$= \frac{1}{p}\left[e^p - e^{-p}\right]$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos np\, dx$$

$$= \frac{1}{p}\int_{-p}^{p} e^x c\,\text{os}\, np\, dx$$

We know that $\int e^{ax}\cos bx dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$$\therefore a_n = \frac{1}{p}\left[\frac{e^x(\cos nx + n\sin nx)}{1^2+n^2}\right]_{-p}^{p}$$

$$= \frac{1}{p}\left[\frac{e^x\cos np - e^{-x}\cos nx)}{1^2+n^2}\right] = \frac{(-1)^n(e^x - e^{-x})}{p(1+n^2)}$$

(as $\sin np = 0 = \sin(-np)$ and $\cos np = (-1)^n$))

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin np\, dx$$

$$= \frac{1}{p}\int_{-p}^{p} e^x \sin np\, dx$$

We know that $\int e^{ax}\sin bx dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$$\therefore b_n = \frac{1}{p}\left[\frac{e^x(\sin nx - n\cos nx)}{1^2+n^2}\right]_{-p}^{p}$$

$$= \frac{-n}{p}\left[\frac{e^p\cos np - e^{-p}\cos(-nx)}{1^2+n^2}\right] = \frac{-n}{p}\left[\frac{e^p(-1)^n - e^{-p}(-1)^n}{p(1+n^2)}\right]$$

$$= \frac{-n(-1)^n(e^p - e^{-p})}{p(1+n^2)}$$

$\therefore$ Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{1}{2p}(e^p - e^{-p}) + \sum_{n=1}^{\infty}\frac{(-1)^n(e^p - e^{-p})}{p(1+n^2)}\cos np$$

$$+ \sum_{n=1}^{\infty}\frac{-n(-1)^n(e^p - e^{-p})}{p(1+n^2)}\sin np$$

i.e., $f(x) = \dfrac{e^p - e^{-p}}{2p}\left[1 + \sum_{n=1}^{\infty}\dfrac{(-1)^n 2\cos np}{1+n^2} + \sum_{n=1}^{\infty}\dfrac{(-1)^n 2\sin np}{1+n^2}\right]$

$$= \frac{\sinh p}{2p}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2\cos np}{1+n^2} - \sum_{n=1}^{\infty}\frac{(-1)^n 2n\sin np}{1+n^2}\right]$$

as $\left(\sinh p = \dfrac{e^p - e^{-p}}{2}\right)$

7) Obtain the Fourier series for $f(x) = x$ in $(-p, p)$ and prove that $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7}\ldots = \dfrac{p}{4}$

Solution :

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$= \frac{1}{p}\int_{-p}^{p} xdx = \frac{1}{p}\left[\frac{x^2}{2}\right]_{-p}^{p} = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} x\cos nxdx$$

$$= \frac{1}{p}\left[x\frac{\sin nx}{n} - 1(\frac{-\cos nx}{n^2})\right]_{-p}^{p}$$

$$= \frac{1}{p}\left[\left(p\,\frac{\sin np}{n} + \frac{\cos np}{n^2}\right) - \left(-p\,\frac{\sin(-np)}{n} + \frac{\cos np}{n^2}\right)\right]$$

$$= \frac{1}{p}\left[0 + \frac{(-1)^n}{n^2} - \frac{(-1)^n}{n^2}\right] = 0$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin nx\,dx$$

$$= \frac{1}{p}\int_{-p}^{p} x\sin nx\,dx$$

$$= \frac{2}{p}\int_{0}^{p} x\sin nx\,dx \quad \because x\sin nx \text{ is even}$$

$$= \frac{2}{p}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^p$$

$$= \frac{2}{p}\left[\left(\frac{-p\cos np}{n}\right) + \left(\frac{\sin np}{n^2}\right) - (0+0)\right]$$

$$= \frac{-2}{p}\frac{p(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= 0 + 0 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2\sin nx}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2\sin nx}{n}$$ is the Fourier series.

Put $x = \frac{p}{2}$ in the Fourier series

$$\therefore f\left(\frac{p}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2\sin\frac{np}{2}}{n}$$

$$= \sum_{n=1,3,5}^{\infty} \frac{(-1)^{n+1} 2\sin\frac{np}{2}}{n}$$ since $\sin\frac{np}{2} = 0$ if n is even.

$$\therefore \frac{p}{2} = 2\left[\frac{\sin\frac{p}{2}}{1} + \frac{\sin\frac{3p}{2}}{3} + \frac{\sin\frac{5p}{2}}{5} + \dots\right]$$

$$\therefore \frac{p}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$ie., 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{p}{4}$$

8) Find the Fourier series for e^{-x} in the interval $(-l, l)$

Solution :

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx$$

$$= \frac{1}{l}\left[e^{-x}\right]_{-l}^{l}$$

$$= -\frac{1}{l}\left[e^{-l} - e^{l}\right]$$

$$= \frac{e^{-l} - e^{l}}{l} = \frac{2\sin hl}{l}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{np\,x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\frac{np\,x}{l}\,dx$$

$$= \frac{1}{l}\frac{\left[e^{-x}\left(-\cos\frac{np\,x}{l} + \frac{p\,n}{l}\sin\frac{np\,x}{l}\right)\right]_{-l}^{l}}{(-1)^2 + \left(\frac{np}{l}\right)^2}$$

$$=\frac{1}{l}\frac{\left[e^{-l}(-\cos np+\frac{np}{l}\sin np)-e^{l}(-\cos np-\frac{np}{l}\sin np)\right]}{(-1)+\left(\frac{np}{l}\right)^2}$$

$$=\frac{1}{l}\left[\frac{(-1)^n(e^l-e^{-l})}{l^2+n^2p^2}\right]l$$

$$=\frac{1}{l^2+n^2p^2}(-1)^n(e^l-e^{-l})=\frac{l(-1)^n 2\sinh l}{l^2+n^2p^2}$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{np\,x}{l}dx$$

$$=\frac{1}{l}\int_{-l}^{l} e^{-x}\sin\frac{np\,x}{l}dx$$

$$=\frac{1}{l}\left[\frac{e^{-l}(-\sin\frac{np\,x}{l}-\frac{np}{l}\cos\frac{np\,x}{l})}{(-1)^2+\left(\frac{np}{l}\right)^2}\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[\frac{e^{-l}(-\sin np-\frac{np}{l}\cos np)-e^{l}(\sin np-\frac{np}{l}\cos np)}{1+\frac{n^2p^2}{l^2}}\right]$$

$$=\frac{1}{l}\frac{\left[e^{-l}\left(\frac{-np}{l}\right)(-1)^n+e^{l}\left(n\frac{p}{l}\right)(-1)^n\right]l^2}{l^2+n^2p^2}$$

$$=\frac{1}{l}\frac{l^2(-1)^n\frac{np}{l}(e^l-e^{-l})}{l^2+n^2p^2}=\frac{(-1)^n np\,2\sinh l}{l^2+n^2p^2}$$

$\therefore$ Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{np\,x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{np\,x}{l}$$

$$\therefore f(x)=\frac{\sinh l}{l}+\sum_{n=1}^{\infty}\frac{(-1)^n 2l\sinh l}{l^2+n^2p^2}\cos\frac{np\,x}{l}$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n 2np\sinh l}{l^2+n^2p^2}\sin\frac{np\,x}{l}$$

i.e., $\therefore f(x)=\frac{\sinh l}{l}[1+\sum_{n=1}^{\infty}\frac{(-1)^n 2l^2}{l^2+n^2p^2}\cos\frac{np\,x}{l}$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n 2np\,l}{l^2+n^2p^2}\sin\frac{np\,x}{l}$$

9) Expand f(x) = x sinx, 0 <x < 2π in a fourier series

$$a_0=\frac{1}{p}\int_0^{2p}x\sin x\,dx=\frac{1}{p}[-x\cos x+\sin x]_0^{2p}=-2$$

$$a_n=\frac{1}{p}\int_0^{2p}x\sin x.\cos nx\,dx$$

$$=\frac{1}{2p}\int_0^{2p}x(\sin(n+1)x-\sin(n-1)x)dx$$

$$=\frac{1}{2p}[x(-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1})$$

$$-\int(-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1})dx]$$

$$=\frac{1}{2p}[2p(-\frac{1}{n+1}+\frac{1}{n-1})]=\frac{2}{n^2-1}$$

$$b_n=\frac{1}{p}\int x\sin x\sin nx\,dx$$

$$=\frac{1}{2p}\int x(\cos(1-n)x-\cos(1+n)x)dx$$

$$= \frac{1}{2p}\left[x(\frac{\sin(1-n)x}{1-n}) - \frac{\sin(1+n)x}{1+n} - \int \frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n} \right] dx$$

$$= \frac{1}{2p}\left[0 + \frac{\cos(1-n)x}{(1-n)^2} - \frac{\cos(1+n)x}{(1+n)^2} \right]_0^{2p} = 0$$

$$f(x) = -1 + \sum \left(\frac{2}{n^2 - 1} \right) \cos nx$$

Note : When $x = \frac{p}{2}$ we derive that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots \infty = \frac{p+2}{4}$$

10) Find the fourier series for the periodic function $f(x) = |x|$ in (-*l*, *l*)

Given $f(x) = |x|$ which is even

$\therefore$ The fourier series is $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos(\frac{np\, x}{l})$

$$a_0 = \frac{2}{l}\int_0^l x dx = \frac{2}{l}\left(\frac{x^2}{2} \right)_0^l = l$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{np\, x}{l} dx$$

$$= \frac{2}{l}\int_0^l x \cos(\frac{np\, x}{l}) dx$$

$$= \frac{2}{l}\left[x.\frac{\sin(\frac{np\, x}{l})}{\frac{np}{l}} - \int \sin \frac{\sin(\frac{np\, x}{l})}{\frac{np}{l}} dx \right]$$

$$= \frac{2}{l}\left[\frac{lx}{np} \sin(\frac{np\, x}{l}) + \frac{l^2}{n^2 p^2} \cos(\frac{np\, x}{l}) \right]_0^l$$

$$= \frac{2}{l}\left[0 + \frac{l^2}{n^2 p^2} \cos(np) - \frac{l^2}{n^2 p^2} \right]$$

$$= \frac{2l}{n^2 p^2}(\cos np - 1) = \frac{2l}{n^2 p^2}((-1)^n - 1)$$

$$\therefore f(x) = \frac{l}{2} + \sum \frac{2l}{n^2 p^2}((-1)^n - 1) \cos\left(\frac{np\, x}{l} \right)$$

11) Expand $f(x) = \begin{cases} x & for \quad 0 \le x < p \\ 2p - x & for \quad p \le x < 2p \end{cases}$ as a fourier series.

$$a_0 = \frac{1}{p}\int_0^p x dx + \frac{1}{p}\int_p^{2p} (2p - x)$$

$$= \frac{1}{p}\left(\frac{p^2}{2} \right) + \frac{1}{p}(2p\, x - \frac{x^2}{2} \Big]_p^{2p}$$

$$= \frac{p}{2} + \frac{1}{p}[(4p^2 - 2p^2) - (2p^2 - \frac{p^2}{2})]$$

$$= \frac{p}{2} + \frac{p}{2} = p$$

$$a_n = \frac{1}{p}\int_0^p x \cos nx dx + \frac{1}{p}\int_p^{2p} (2p - x) \cos nx dx$$

$$= \frac{1}{p}\left[\frac{x \sin(nx)}{n} - \int \frac{\sin(nx)}{n} dx \right]$$

$$+ \frac{1}{p}\left[(2p - x) \frac{\sin nx}{n} + \int \frac{\sin nx}{n} .dx \right]$$

$$= \frac{1}{p}\left[0 + \frac{\cos nx}{n^2} \right]_0^p + \frac{1}{p}\left[0 - \frac{\cos nx}{n^2} \right]_p^{2p}$$

$$= \frac{1}{p\, n^2}\left[(-1)^n - 1 \right] + \frac{1}{p}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2} \right]$$

$$=\frac{2}{p n^2}\left[(-1)^n-1\right]$$

$$b_n=\frac{1}{p}\int_0^p x\sin nx dx+\frac{1}{p}\int_p^{2p}(2p-x)\sin nx dx$$

$$=\frac{1}{p}\left[\frac{-x\cos nx}{n}+\int\frac{\cos nx}{n}dx\right]_0^p$$

$$+\frac{1}{p}\left[(2p-x)\left(\frac{-\cos nx}{n}\right)+\int\frac{\cos nx}{n}.dx\right]_p^{2p}$$

$$=\frac{1}{p}\left[\frac{p\ \cos nx}{n}\right]_0^p+\frac{1}{p}\left[\frac{p\ \cos nx}{n}\right]_p^{2p}=0$$

$\therefore$ The fourier series is

$$f(x)=\frac{p}{2}+\sum\frac{2}{p n^2}((-1)^n-1)\cos nx+0$$

$$=\frac{p}{2}+\sum\frac{2}{p n^2}((-1)^n-1)\cos nx.$$

12) Find a fourier series for the function

$$f(x)=\begin{cases}-1 & -p<x<0\\ 0 & x=0\\ 1 & 0<x<p\end{cases}$$

$$a_0=\frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$=\frac{1}{p}\int_{-p}^{0}-1dx+\frac{1}{p}\int_0^p 1dx=\frac{1}{p}[-p+p]=0$$

$$a_n=\frac{1}{p}\int_{-p}^{p} f(x)\cos(nx)dx$$

$$=\frac{1}{p}\int_{-p}^{0}-\cos nx dx+\frac{1}{p}\int_0^p 1.\cos nx dx$$

$$=\frac{1}{p}\left[-\frac{\sin nx}{n}+\frac{\sin nx}{n}\right]$$

$$=\frac{1}{p}(0)=0$$

$$b_n=\frac{1}{p}\int_{-p}^{p} f(x)\sin(nx)dx$$

$$=\frac{1}{p}\int_{-p}^{0}(-1)\sin nx dx+\frac{1}{p}\int_0^p 1.\sin nx dx$$

$$=\frac{1}{p}\left[\frac{\cos nx}{n}\right]_{-p}^{0}+\left(\frac{-\cos nx}{n}\right)\Bigg]_0^p$$

$$=\frac{1}{np}[1-\cos np-\cos np+1]$$

$$=\frac{2}{p n}(1-(-1)^n)$$

$\therefore$ b_n is zero for n = 2, 4, 6, . . .

and $b_n=\frac{4}{p n}$ for n = 1, 3, 5, . .

$\therefore$ Required fourier series

$$f(x)=0+\sum 0.\cos nx+\sum\frac{4}{p n}.\sin nx$$

$$=\frac{4}{p}\left[\sin x+\frac{\sin 3x}{3}+\ldots\infty\right]$$

Note : when $x=\frac{p}{2}$

$$f(x)=1=\frac{4}{p}\left[1-\frac{1}{3}+\frac{1}{5}-+\ldots\infty.\right]$$

$$\frac{p}{4}=1-\frac{1}{3}+\frac{1}{5}-+\ldots$$

13) Find the Fourier series for $\sqrt{1-\cos x}$ in the interval

$-\pi < x < \pi$

Let $f(x)=\sqrt{1-\cos x}$. It is an even function

$$\therefore f(x)=\frac{a_0}{2}+\sum a_n\cos nx; \quad b_n=0$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}\sqrt{1-\cos x}\,dx=\frac{2}{\pi}.\sqrt{2}\int\sin\frac{x}{2}dx$$

$$=\frac{2\sqrt{2}}{\pi}\left[-\frac{\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi}=\frac{4\sqrt{2}}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sqrt{1-\cos x}.\cos nx\,dx=\frac{2\sqrt{2}}{\pi}\int_0^{\pi}\sin\frac{x}{2}.\cos nx\,dx$$

$$=\frac{2\sqrt{2}}{\pi}\int_0^{\pi}\frac{1}{2}\{\sin(\frac{x}{2}+nx)+\sin(\frac{x}{2}-nx)\}dx$$

$$=\frac{\sqrt{2}}{\pi}\left[-\frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}}+\frac{\cos(n-\frac{1}{2})x}{n-\frac{1}{2}}\right]_0^{\pi}$$

$$=\frac{\sqrt{2}}{\pi}\left[(0+0)+\frac{1}{n+\frac{1}{2}}-\frac{1}{n-\frac{1}{2}}\right]$$

$$=\frac{\sqrt{2}}{\pi}\left[\frac{n-\frac{1}{2}-n-\frac{1}{2}}{n^2-\frac{1}{4}}\right]$$

$$=-\frac{4\sqrt{2}}{\pi}\left(\frac{1}{4n^2-1}\right)$$

$$\therefore f(x)=\frac{2\sqrt{2}}{\pi}-\frac{4\sqrt{2}}{\pi}\sum\left(\frac{1}{4n^2-1}\right)\cos nx$$

14) If α is not an integer show that for $-\pi < x < \pi$

$$\sin\alpha x=\frac{2\sin\alpha\pi}{\pi}\left[\frac{\sin x}{1^2-\alpha^2}-\frac{2\sin 2x}{2^2-\alpha^2}+\frac{3\sin 3x}{3^2-\alpha^2}-+\ldots\right]$$

Since $f(x)=\sin\alpha x$ is an odd function, a_0 & a_n are equal to zero.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin\alpha x.\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{1}{2}(\cos(n-\alpha)x-\cos(n+\alpha)x)dx$$

$$=\frac{1}{\pi}\left[\frac{\sin(n-\alpha)x}{n-\alpha}-\frac{\sin(n+\alpha)x}{n+\alpha}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{\cos n\pi\sin\alpha\pi}{n-\alpha}-\frac{\cos n\pi\sin\alpha\pi}{n-\alpha}\right]$$

$$=-\frac{\cos n\pi\sin\alpha\pi}{\pi}\left(\frac{2n}{n^2-\alpha^2}\right)$$

$$\therefore\sin\alpha x=\sum\frac{-\cos n\pi\sin\alpha\theta}{\pi}\left(\frac{2n}{n^2-\alpha^2}\right)\sin nx$$

$$=\frac{2\sin\alpha\pi}{\pi}\sum\frac{-n\cos n\pi.\sin\alpha x}{n^2-\alpha^2}$$

$$=\frac{2\cos n\pi\sin\alpha\pi}{\pi}\left[\frac{\sin x}{1^2-\alpha^2}-\frac{2\sin 2x}{2^2-\alpha^2}+\frac{3\sin 3x}{3^2-\alpha^2}-+\ldots\right]$$

Exercise :

I A.

1. Define a Fourier series
2. Write the empherical formulae for the fourier coefficients.
3. Write the fourier series with period 2π in the interval (c, c + 2π)
4. Derive the Euler's formulae in the interval (c, c + 2l)

5. Write the Fourier coefficients in the interval $(-l, l)$ when f(x) is a) even and b) odd.
6. Mention dirichlets conditions.
7. Find the fourier coefficient a_0 for the following functions : -
   - (i) $f(x) = x^2$ in $-\pi < x < \pi$
   - (ii) $f(x) = x^2$ in $-l < x < l$
   - (iii) $f(x) = \begin{cases} x & 0 < x < p \\ 2p - x & p < x < 2p \end{cases}$
   - (iv) $f(x) = |x| \quad -p < x < p$
   - (v) $f(x) = \cos l\,x$ in $-p \le x \le p$
   - (vi) $f(x) = \begin{cases} -1 & -p < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < p \end{cases}$
   - (vi) $f(x) = \begin{cases} 1 + \dfrac{2x}{p} & -p < x \le 0 \\ 1 - \dfrac{2x}{p} & 0 < x < p \end{cases}$
8. Find the fourier coefficients a_n and b_n for the above problems.

(2 marks for each constants)

B. 1. Find the Fourier series for

a) $f(x) = x^2$ in $-p < x < p$. Hence deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{p^2}{8}$$

b) $f(x) = \begin{cases} -x & in \quad -p < x \le 0 \\ x & in \quad 0 < x < p \end{cases}$

Hence deduce $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} \ldots \infty = \dfrac{p}{4}$

c) $f(x) = |x|$ in $-p < x < p$. Hence deduce

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots \infty = \frac{p^2}{8}$$

d) $f(x) = |\sin x|$

e) $f(x) = 1 + \dfrac{2x}{p}, \quad -p \le x \le 0$

$= 1 - \dfrac{2x}{p}, \quad 0 \le x \le p$

f) $f(x) = \cos ax$ in $-p \le x \le p$, a is not an integer.

g) $f(x) = \begin{cases} x + \dfrac{p}{2} & in \quad -p < x \le 0 \\ \dfrac{p}{2} - x & in \quad 0 \le x < p \end{cases}$

h) $f(x) = x(p - x) \quad 0 \le x \le p$

If $f(x) = \begin{cases} x & in \quad 0 < x < p/2 \\ p - x & in \quad p/2 < x < p \end{cases}$

j) $f(x) = \begin{cases} p\,x & in \quad 0 \le x \le 1 \\ p\,(2 - x) & in \quad 1 \le x \le 2 \end{cases}$

k) $f(x) = \begin{cases} x & in \quad (0, l) \\ p - 2l & in \quad (l, 2l) \end{cases}$

l) $f(x) = \begin{cases} -x^2 & -p < x < 0 \\ x^2 & 0 < x < p \end{cases}$

m) $f(x) = x^3$ in $-p < x < p$

n) $f(x) = \begin{cases} 1 & -p < x < 0 \\ 0 & 0 < x < p \end{cases}$

o) $f(x) = \begin{cases} 1 & -p \le x < 0 \\ 2 & 0 < x \le p \end{cases}$

p) $f(x)=\begin{cases}-a & -p<x<0\\ a & 0<x<p\end{cases}$

q) $f(x)=\begin{cases}p+x & -p<x<0\\ p-x & 0<x<p\end{cases}$

r) $f(x)=\frac{x^2}{4},\quad -p<x<p$, Hence $\frac{p^2}{6}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots\infty$

II. 1 . Show that the fourier series for f(x) = 1-x^2 in (-1, 1) is

$$=\frac{2}{3}-\frac{4}{p}\sum\frac{(-1)^n}{n}\cos p\,x$$

(Hint f(x) is even)

2. Show that the fourier series of

$$f(x)=\begin{cases}x & in\ -p/2<x\le p/2\\ p-x & in\ p/2<x<3p/2\end{cases}\quad \text{is}$$

$$f(x)=\frac{4}{p}\sum\frac{(-1)^n}{n}\sin(2n+1)x$$

3. Show that the fourier series of

$$f(x)=\begin{cases}-\frac{p+x}{2} & -p<x<0\\ \frac{p-x}{2} & in\ 0<x<p\end{cases}\quad \text{is}$$

$$f(x)=\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\infty$$

(Hint : f(x) is odd)

4. Show that the fourier series of

$f(x)=|\cos x|$ in $(-p,p)$ is

$$f(x)=\frac{2}{p}+\frac{4}{p}\left(\frac{1}{3}\cos 2x-\frac{1}{15}\cos 4x+\ldots\infty\right)$$

5. If $f(x)=x+x^2$ for $-p<x<p$, show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{p^2}{6}\ \text{and}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{p^2}{8}$$

6. If f(x) =x in $(-p,p)$, show that

$$f(x)=2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots\right)$$

(Hint f(x) is odd)

Answers

A. 7 (i) $\frac{2p^2}{3}$ (ii) $\frac{2l^2}{3}$ (iii) p (iv) p (v) $\frac{2\sin lp}{lp}$

8. (i) $a_n=\frac{4l^2(-1)^n}{n^2p^2},\quad b_n=0$ (ii) $a_n=(-1)^n\frac{4}{n^2};\quad b_n=0$

(iii) $a_n=\frac{2}{n^2p}((-1)^n-1),\quad b_n=0$ (iv) $a_n=\frac{-4}{p(2m-1)^2},\quad b_n=0$

(v) $a_n=\frac{(-1)^n l\sin lp}{l^2-n^2},\quad b_n=0$ (vi) $a_n=0,\quad b_n=\frac{2}{np}[1-(-1)^n]$

(vii) $a_n=\frac{4}{n^2p^2}[1-(-1)^n],\quad b_n=0$

B. I

d) $f(x)=\frac{2}{p}-\frac{4}{p}\left\{\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\frac{\cos 6x}{35}+\ldots\infty\right\}$

e) $f(x)=\frac{8}{p^2}\left\{\cos x+\frac{1}{9}\cos 3x+\frac{1}{25}\cos 5x+\ldots\infty\right\}$

f) $f(x)=\frac{1}{p}\left\{\frac{1}{a}-\sum_{1}^{\infty}\frac{2a}{n^2-a^2}\right\}$

g) $f(x)=\frac{4}{p}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 3x}{5^2}+\ldots\infty\right)$

h) $f(x) = -\pi^2 - 8\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots\right)$

$$+\frac{2}{\pi}\left[\left(\frac{3\pi^2}{1} - \frac{4}{1^2}\right)\sin x + \frac{\pi^2 \sin 2x}{2} + \left(\frac{3\pi^2}{1} - \frac{4}{1^2}\right)\sin x + \frac{\pi^2 \sin 2x}{2} + \dots\right]$$

i) (i) $f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - + \dots\right]$

(ii) $f(x) = \frac{4}{\pi} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots\infty\right]$

j) $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 3\pi x}{5^2} + \dots\infty\right]$

k) $\frac{2l}{\pi}\sum \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$

l) $\frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^2}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^2}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x + \dots\right]$

m) $2\left[\left(\frac{\pi^2}{1} - \frac{6}{1^3}\right)\sin x - \left(\frac{\pi^2}{2} - \frac{6}{2^3}\right)\sin 2x + \left(\frac{\pi^2}{3} - \frac{6}{3^3}\right)\sin 3x \dots\right]$

n) $f(x) = \frac{1}{2} - \frac{2}{\pi}\sum \frac{\sin(2n-1)x}{2n-1}$

o) $f(x) = \frac{3}{2} + \frac{2}{\pi}\sum \frac{\sin(2n-1)x}{2n-1}$

p) $f(x) = \frac{4a}{\pi}\sum \frac{\sin(2n-1)x}{2n-1}$

q) $f(x) = \frac{3\pi}{8} + \frac{2}{\pi}\sum (1-(-1)^n)\frac{1}{n^2}\cos nx$

5.8 Half – range cosine and sine series

Many times, it may be required to obtain a Fourier series expansion of a function in the interval $(0, l)$ which is half the period of the Fourier series. This is achieved by treating $(0, l)$ as half – range of $(-l, l)$ and defining f(x) suitably in the other half i.e., in $(-l, 0)$ so as to make the function even or odd according as cosine series or sine series is required.

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)dx,\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ for half – range cosine series and

$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$ and write the series as

$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ for half – range sine series.

Similarly, in $(0, \pi)$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

and $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

NOTE : (i) To solve a problem on Fourier series we have to find a_0, a_n and b_n and substitute in

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

(ii) Finding of a_0, a_n, b_n, involves integration. In most of the problems, f(x) consists of terms like x, x^2, x^3, etc which after a few differentiation will be zero.

The generalized formula for integration of the product of two functions u and v called the Bernoulli's rule may be used for finding a_n and b_n.

$$\int uv dx = uv^1 - u'v^2 + u''v^3 - u'''v^4 + - \dots$$

where dashes denote differentiation w.r.t x and suffixes 1,2,3,. . . denote integration w.r.t. x

For eg. $\int x^2 \sin nx dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$

(iii) The following values of cosine and sine are useful

$\cos 0 = 1$, $\cos n\pi = (-1)^n = \cos(-n\pi)$, $\cos\frac{np}{2} = 0$ if n is odd and

$\cos\frac{np}{2} = (-1)\frac{n}{2}$ if n is even.

$\sin 0 = 0$, $\sin n\pi = \sin(n\pi)$, $\sin\frac{np}{2} = (-1)^{\frac{n-1}{2}}$ if n is odd and

$\sin\frac{np}{2} = 0$ is even.

(iv) Integration work can be reduced to a great extent by using the ideas of even and odd functions, whenever 0 is the mid point.

(v) If f(x) is neither odd nor even, then f(x) may consist of some terms which when taken individually may be odd or even and the integration work can be reduced.

Worked Examples :

1) Find the half range sine series for f(x) =x in (0, 1)

(May 2003)

$f(x) = \sum b_n \sin\left(\frac{np\,x}{L}\right)$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{np\,x}{L}\right)dx$

$$b_n = \frac{2}{1}\int_0^1 x.\sin np\,xdx$$

$$= 2\left[-\frac{x\cos np\,x}{np} + \frac{1}{np}\int \cos np\,xdx\right]$$

$$= 2\left[-\frac{x\cos np\,x}{np} + \frac{\sin np\,x}{(np)^2}\right]_0^1$$

$$= 2\left[-\frac{x\cos np\,x}{np}\right] = \frac{2(-1)^n}{np}$$

$\therefore$ Half = ramge Sine series is

$$= \sum_1^{\infty}\frac{-2(-1)^n}{np}.\sin(np)x$$

2) Obtain the half -range Sine series for f(x) = x over the interval (0, π) (A 2003)

$$b_n = \frac{2}{p}\int_0^p x.\sin nxdx$$

$$= \frac{2}{p}\left[x(-\frac{\cos nx}{n}) + \frac{1}{n}\int \cos nxdx\right]$$

$$= \frac{2}{p}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^p$$

$$= \frac{2}{p}\left[-\frac{p\cos np}{n}\right] = \frac{2(-1)^n}{n}$$

$\therefore$ Half = ramge Sine series is

$$f(x) = \sum\frac{-2(-1)^n}{n}.\sin nx.$$

3) Find the half – range Fourier sine series of f(x) = x^2 in the interval (0, 1) (N 2000)

$$b_n = \frac{2}{1}\int_0^1 x^2\sin np\,xdx$$

$$= 2\left[x^2(-\frac{\cos np\,x}{np}) + \frac{1}{np}\int\cos(np\,x).2xdx\right]$$

$$= 2\left[-\frac{\cos np\,x}{np} + \frac{2}{np}\left(x\frac{\sin np\,x}{np} - \int\frac{\sin np\,x}{np}.dx\right)\right]$$

$$= 2\left[-\frac{\cos np\,x}{np} + \frac{2}{np}\left(0 + \frac{\cos np\,x}{n^2p^2}\right)_0^1\right]$$

$$= 2\left[-\frac{\cos np\,x}{np} + \frac{2}{np}\left(\frac{\cos np\,x}{n^2p^2} - \frac{1}{n^2p^2}\right)\right]$$

$$\therefore f(x) = \sum 2\left(\frac{(-1)^n}{np} + \frac{2(-1)^n}{n^3p^3} - \frac{2}{n^3p^3}\right)\sin(np\,x)$$

4) Find the half range cosine series for the function f(x) =x^2 in (0,π) (A 2003)

It is required to find

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos\left(\frac{np\,x}{L}\right) \text{ where}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx; \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{np\,x}{L}\right)dx$$

$$a_0 = \frac{2}{p}\int_0^p x^2dx = \frac{2}{p}\left[\frac{x^3}{3}\right]_0^p = \frac{2p^2}{3}$$

$$a_0 = \frac{2}{p}\int_0^p \cos(nx)dx$$

$$= \frac{2}{p}\left[x^2\frac{\sin(nx)}{n} - \int\frac{\sin(nx)}{n}.2xdx\right]$$

$$= \frac{2}{p}\left[0 - \frac{2}{n}\left\{x\left(-\frac{\cos(nx)}{n}\right) + \int\frac{\cos(nx)}{n}.1dx\right\}\right]$$

$$= \frac{2}{p}\left[\frac{2}{n}\left(-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right)\right]_0^p$$

$$= \frac{2}{p}\left[-\frac{2}{n}\left(-\frac{p(-1)^n}{n}\right)\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{2p^2}{2(3)} + \sum\frac{4(-1)^n}{n^2}\cos(nx)$$

$$= \frac{p^2}{3} + \sum\frac{4(-1)^n}{n^2}\cos(nx)$$

5) Find the half range Sine series for $f(x) = p\,x - x^2$ in the internal $0<x<p$

$$b_n = \frac{2}{p}\int_0^p (p\,x - x^2)\sin xdx$$

$$= \frac{2}{p}\int_0^p (p\,x - x^2)(-\frac{\cos nx}{n}) - (p - 2x)(-\frac{\sin nx}{n^2}) + (-2)(-\frac{\cos nx}{n})\Big]_0^p$$

$$= \frac{4}{p\,n^3}(1 - \cos np)$$

$$= \frac{8}{p\,n^3}$$

$$\therefore f(x) = \sum\frac{8}{p\,n^3}\sin nx$$

$$= \frac{8}{p}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right]$$

6) Find half – range sine series of

$$f(x) = \begin{cases} x & 0 < x \le p/2 \\ p - x & p/2 < x < p \end{cases}$$

$$b_n = \frac{2}{p}\int_0^p f(x)\sin xdx$$

$$= \frac{2}{p}\int_0^{p/2} x\sin xdx + \frac{2}{p}\int_{p/2}^{p/2}(p - x)\sin xdx$$

$$= \frac{2}{p}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{p/2}$$

$$+\frac{2}{p}\left[(p-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{p/2}^{p}$$

$$=\frac{2}{p}\left[-\frac{p}{2}\frac{\cos(np/2)}{n}+-\frac{\sin(np/2)}{n^2}+\frac{p}{2}\frac{\cos(np/2)}{n}+\frac{\sin np/2}{n^2}\right]$$

$$=\frac{4}{p\,n^2}\sin\left(\frac{np}{2}\right)$$

$$\therefore f(x)=\frac{4}{p}\left[\sin x-\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}-+\ldots\infty\right]$$

7) Find the half – range sine series for f(x) = 2x-1 in the interval (0, 1) (A 2001)

$$b_n=\frac{2}{1}\int_0^1(2x-1)\sin(np\,x)dx$$

$$=2\left[(2x-1)\left(\frac{-\cos np\,x}{np}\right)-(2)\left(\frac{-\sin(np\,x)}{n^2p^2}\right)\right]_0^1$$

$$=2\left[\frac{-\cos np}{np}-\frac{1}{np}\right]$$

$$\therefore f(x)=\sum-\frac{2}{np}(1+\cos np)\sin(nxp)$$

8. Find the half – range cosine series for the function of f(x) = (x - 1)^2 in the interval 0 < x < 1.

$$a_0=\frac{2}{1}\int_0^1(x-1)^2dx=\left.\frac{2(x-1)^3}{3}\right]_0^1=0+2/3=2/3$$

$$a_n=\frac{2}{1}\int_0^1(x-1)^2.\cos\left(\frac{np\,x}{l}\right)dx$$

$$=2\int_0^1(x-1)^2\cos(np\,x)dx$$

$$=2\left[(x-1)^2\left(\frac{\sin np\,x}{np}\right)-2(x-1)\left(-\frac{\cos np\,x}{n^2p^2}\right)+2\left(\frac{-\sin np\,x}{n^3p^3}\right)\right]_0^1$$

$$=2\left[\frac{-2\sin(np)}{n^2p^2}+-\frac{2\cos(0)}{n^2p^2}\right]=\frac{4}{n^2p^2}$$

$$f(x)=\frac{1}{3}+\sum\frac{4}{n^2p^2}\cos np\,x$$

9) Expand f(x) = x as a cosine half – range series in 0 < x <2

Solution : The graph of f(x) = x is a straight line. Let us extend the function f(x) in the interval (-2, 0) so that the new function is symmetric al about they y – axis and hence it represents an even function in (-2, 2)

$\therefore$ the Fourier coefficient $b_n = 0$

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{np\,x}{2}$$

$$a_0=\frac{1}{2}\int_{-2}^{2}x\,dx=\frac{1}{2}2\int_{-2}^{2}x\,dx=\left[\frac{x^2}{2}\right]_0^2=\frac{4}{2}=2$$

$$a_n=\frac{2}{2}\int_{-2}^{2}x\cos\frac{np\,x}{2}dx$$

$$=\left[x\frac{\sin\frac{np\,x}{2}}{\frac{np}{2}}-1\frac{-\cos\frac{np\,x}{2}}{\left(\frac{np}{2}\right)^2}\right]_0^2$$

$$=\left[\frac{2x}{np}\sin\frac{np\,x}{2}+\frac{4}{n^2p^2}\cos\frac{np\,x}{2}\right]_0^2$$

$$= \left(0+\frac{4}{n^2p^2}\cos np\right)-\left(0+\frac{4}{n^2p^2}\cos 0\right)$$

$$=\frac{4}{n^2p^2}(\cos np-1)=\frac{4}{n^2p^2}[(-1)^n-1]$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{np x}{2}$$

$$=\frac{2}{2}+\sum_{n=1}^{\infty}\frac{4}{n^2p^2}[(-1)^n-1]\cos\frac{np x}{2}$$

$$f(x)=1+\frac{4}{p^2}\left[\frac{-2\cos\frac{p x}{2}}{1^2}+0+\frac{-2\cos\frac{3p x}{2}}{3^2}+0+\frac{-2\cos\frac{5p x}{2}}{5^2}+\ldots\right]$$

$$i.e., f(x)=1-\frac{8}{p^2}\left[\frac{\cos\frac{p x}{2}}{1^2}+\frac{\cos\frac{3p x}{2}}{3^2}+\frac{\cos\frac{5p x}{2}}{5^2}+\ldots\right]$$

Important Note : It must be clearly understood that we expand a function in $0 < x < c$ as a series of sines and cosines merely looking upon it as an odd or even function of period 2c. It hardly matters whether the function is odd or even .

10) Expand $f(x)=\frac{1}{4}-x$ if $0<x<\frac{1}{2}$

$$= x-\frac{3}{4} \text{ if } \frac{1}{2}<x<1$$

in the Fourier series of sine terms

Solution : Let f(x) be an odd function in (-1, 1)

$\therefore$ $a_0 = 0$ and $a_n = 0$

and $b_n = 1\int_{-1}^{1} f(x)\sin\frac{np x}{1}dx$

$$=2\int_0^1 f(x)\sin np x\,dx$$

$$=2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin np x\,dx+\int_{1/2}^{1}\left(x-\frac{3}{4}\right)\sin np x\,dx\right]$$

$$=2\left[\left(\frac{1}{4}-x\right)\left(\frac{-\cos np}{np}\right)-(-1)\left(\frac{-\sin np x}{n^2p^2}\right)\right]_0^{1/2}$$

$$+2\left[\left(x-\frac{3}{4}\right)\left(\frac{-\cos np}{np}\right)-(-1)\left(\frac{-\sin np x}{n^2p^2}\right)\right]_{1/2}^{1}$$

$$=2\left[\frac{1}{4np}\cos\frac{np}{2}-\frac{\sin\frac{np}{2}}{n^2p^2}+\frac{1}{4np}\cos 0+0\right]$$

$$+2\left[-\frac{1}{4np}\cos np+0-\frac{1}{4np}\cos np-\frac{\sin\frac{np}{2}}{n^2p^2}\right]$$

$$i.e., b_n=\frac{1}{2np}[1-(-1)^n]-\frac{4\sin\frac{np}{2}}{n^2p^2} \text{ since } \cos\frac{np}{2}=0$$

$\therefore b_1=\frac{1}{p}-\frac{4}{p^2};\quad b_2=0$

$b_3=\frac{1}{3p}+\frac{4}{3^2p^2};\quad b_4=0$

$b_5=\frac{1}{5p}-\frac{4}{5^2p^2};\quad b_6=0$ etc.

$$\therefore f(x)=\sum_{n=1}^{\infty} b_n\sin np x$$

$$\left(\frac{1}{p}-\frac{4}{p^2}\right)\sin p x+\left(\frac{1}{3p}+\frac{4}{3^2p^2}\right)\sin 3p x$$

$$+\left(\frac{1}{5p}-\frac{4}{5^2p^2}\right)\sin 5p x+\ldots$$

11) Find the sine and cosine series of the function $f(x) = p - x$ in $0 < x < p$. (A 99)

Solution :

(i) Fourier sine series:

$$b_n = \frac{2}{p}\int_0^{\infty} f(x)\sin nx dx$$

$$= \frac{2}{p}\int_0^{\infty} (p - x)\sin nx dx$$

$$= \frac{2}{p}\left[(p-x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n}\right)\right]_0^x$$

$$= \frac{2}{p}\left[(0-0) - (p-0)\left(\frac{-\cos 0}{n}\right) + \left(\frac{\sin 0}{n}\right)\right]$$

$$= \frac{2}{p}\frac{p}{n} = \frac{2}{n}$$

$\therefore$ Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\frac{2}{n}\sin nx$$

$$= 2\left[\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

(ii) Fourier cosine series:

$$a_0 = \frac{2}{p}\int_0^{p} f(x)dx = \frac{2}{p}\int_0^{p}(p - x)dx$$

$$= \frac{2}{p}\left[p x - \frac{x^2}{2}\right]_0^p$$

$$= \frac{2}{p}\left[p^2 - \frac{p^2}{2}\right] = \frac{2}{p}.\frac{p^2}{2} = p$$

$$a_0 = \frac{2}{p}\int_0^{p}(p - x)\cos nx dx$$

$$= \frac{2}{p}\left[(p-x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^p$$

$$= \frac{2}{p}\left[\left(0.\frac{\cos np}{n}\right) - \left(0 - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{p}\left[\frac{1}{n^2} - \frac{\cos np}{n^2}\right]$$

$$= \frac{2}{n^2 p}(1 - \cos np)$$

$$= \frac{2}{n^2 p}[1 - (-1)^n]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{p}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2 p}[1 - (-1)^n]\cos nx$$

$$= \frac{p}{2} + \frac{2}{p}\left[\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \ldots\right]$$

$$= \frac{p}{2} + \frac{4}{p}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]$$

12) Find the Fourier series expansion with period 3 to represent the function $f(x) = 2x - x^2$ in the range (0, 3)

Solution : We have $c = 0$ and $2l = 3$

$$\therefore a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}[9-9] = 0$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{np\,x}{l}\,dx$$

$$= \frac{2}{3}\int_c^3 (2x-x^2)\cos\frac{2np\,x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{2np\,x}{3}}{\frac{2np\,x}{3}} - (2-2x)\left[\frac{-\cos\frac{2np\,x}{3}}{\left(\frac{2np\,x}{3}\right)^2}\right] + (+2)\left[\frac{-\sin\frac{2np\,x}{3}}{\left(\frac{2np\,x}{3}\right)^2}\right]\right]_0^3$$

$$= \frac{2}{3}\left[-\frac{9}{2np}\sin 2np - \frac{36}{2n^2p^2}\cos 2np + \frac{27}{2n^3p^3}\sin 2np\right]$$

$$-\frac{2}{3}\left[0 + \frac{9}{2n^2p^2} + 0\right]$$

$$= \frac{2}{3}\left[-\frac{9}{n^2p^2} - \frac{3}{n^2p^2}\right] = \frac{2}{3}\left[-\frac{12}{n^2p^2}\right]$$

$$= -\frac{8}{n^2p^2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{np\,x}{l}\,dx$$

$$= \frac{2}{3}\int_c^{c+2l} (2x-x^2)\sin\frac{2np\,x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2np\,x}{3}}{\frac{2np\,x}{3}}\right) + (2-2x)\frac{\sin\frac{2np\,x}{3}}{\left(\frac{2np\,x}{3}\right)^2} + (+2)\frac{\cos\frac{2np\,x}{3}}{\left(\frac{2np\,x}{3}\right)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{9\cos 2np}{2np} - \frac{(-4)9}{2n^2p^2}\sin 2np + \frac{2(27)}{8n^3p^3}\cos 2np\right]$$

$$= \frac{2}{3}\left[0 + 0 + 2\times\frac{27}{8n^3p^3}\cos 0\right]$$

$$= \frac{2}{3}\left[\frac{9}{2np} + \frac{27}{8n^3p^3} - \frac{27}{8n^3p^3}\right] = \frac{3}{np}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{np\,x}{l} + \sum_{n=1}^{\infty} b_n\sin\frac{np\,x}{l}$$

$$= 0 + \sum_{n=1}^{\infty}\frac{-8}{n^2p^2}\cos\frac{2np\,x}{3} + \sum_{n=1}^{\infty}\frac{3}{np}\sin\frac{2np\,x}{3}$$

$$\therefore f(x) = -\frac{8}{p^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2np\,x}{3} + \frac{3}{p}\sum_{n=1}^{\infty}\frac{3}{n}\sin\frac{2np\,x}{3}$$

13) If $f(x) = \left(\frac{p-x}{2}\right)^2$, show that $f(x) = \frac{p^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ in the range of $(0, 2p)$

Solution : It is an even function $\therefore b_n = 0$

$$a_n = \frac{1}{p}\int_0^{2p} f(x)\,dx = \frac{1}{p}\int_0^{2p}\left(\frac{p-x}{2}\right)^2 dx$$

$$= \frac{1}{4p}\left[\frac{p-x}{3(-1)}\right]_0^{2p}$$

$$= -\frac{1}{12p}\left[(-p)^3 - p^3\right]$$

$$= \frac{-1}{12p}(-2p^{3}) = \frac{p^{2}}{6}$$

$$a_n = \frac{1}{p}\int_{0}^{2p} f(x)\cos nxdx$$

$$= \frac{1}{p}\int_{0}^{2p}\left(\frac{p-x}{2}\right)^{2} \cos nxdx$$

$$= \frac{1}{p}\left[\left(\frac{p-x}{2}\right)^{2}\frac{\sin nx}{n}+\frac{(p-x)}{2}\left(\frac{-\cos nx}{n^{2}}\right)\right]+\left[\left(-\frac{1}{2}\right)\left(\frac{-\sin nx}{n^{3}}\right)\right]_{0}^{2p}$$

$$= \frac{1}{p}\,\frac{2p}{2n^{2}} = \frac{p}{n^{2}}$$

$\therefore$ The Fourier series is

$$\therefore f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin npx$$

$$= \frac{p^{2}}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2}}+0$$

$$\therefore f(x) = \frac{p^{2}}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^{2}}$$

14) Find the fourier Series expansion of cosh ax in(-π,π)

Solution : f(x) = cosh ax

$$a_0 = \frac{1}{p}\int_{-p}^{p}\cosh axdx$$

$$= \frac{2}{p}\int_{-p}^{p}\cosh axdx = \frac{2}{p}\left[\frac{\sinh ax}{a}\right]_{0}^{p}$$

$$= \frac{2}{p\,a}\sinh ap$$

$$a_0 = \frac{1}{p}\int_{-p}^{p}\cosh ax\cos nxdx$$

$$= \frac{2}{p}\int_{0}^{p}\frac{e^{ax}+e^{-ax}}{2}\cos nxdx$$

$$= \frac{2}{2p}\left[\int_{0}^{p} e^{ax}\cos nxdx+\int_{0}^{p} e^{-ax}\cos nxdx\right]$$

$$= \frac{1}{p}\left[e^{ax}\left(\frac{a\cos nx+n\sin nx}{a^{2}+n^{2}}\right)+e^{-ax}\left(\frac{a\sin nx-a\cos nx}{a^{2}+n^{2}}\right)\right]_{0}^{p}$$

$$= \frac{1}{p}e^{ax}\left(\frac{a\cos nx+n\sin nx}{a^{2}+n^{2}}\right)+e^{-ax}\left(\frac{a\sin nx-a\cos nx}{a^{2}+n^{2}}\right)$$

$$-e^{0}\left(\frac{a\cos nx+n\sin nx}{a^{2}+n^{2}}\right)-e^{0}\left(\frac{n\sin 0-a\cos 0}{a^{2}+n^{2}}\right)$$

$$= \frac{1}{p}\left[\frac{e^{ap}a}{a^{2}+n^{2}}(-1)^{n}-\frac{e^{-ap}a(-1)^{n}}{a^{2}+n^{2}}-\frac{1}{a^{2}+n^{2}}+\frac{1}{a^{2}+n^{2}}\right]$$

$$= \frac{1}{p}\,\frac{a(-1)^{n}}{a^{2}+n^{2}}(e^{ap}-e^{-ap})$$

$$i.e., a_n = \frac{2a(-1)^{n}}{p\,(a^{2}+n^{2})}\sinh ap$$

$b_n = 0$

\ The Fourier series for cosh ax is

$$\therefore \cosh ax = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos np$$

$$= \frac{1}{p\,a}\sinh p\,a+\sum_{n=1}^{\infty}\frac{2a(-1)^{n}}{p\,(a^{2}+n^{2})}\sinh ap\cos np$$

Exercise

1. Find the half – range Fourier cosine series for f(x) = x in $0 < x \le p$

2. Prove that $f(x)=\begin{cases}\frac{1}{4}-x & 0<x<\frac{1}{2}\\ x-\frac{3}{4} & \frac{1}{2}<x<1,\end{cases}$

the sine series is = $=\sum\left(\frac{1}{2np}[1-(-1)^n]-\frac{4\sin\left(\frac{np}{2}\right)}{n^2p^2}\right)\sin npx$

3. Find the half – range Fourier sine series for
   f(x) = e^x in the interval (0, 1)

4. Find the half – range cosine series for

$$f(x)=\begin{cases}x & 0<x<\frac{a}{2}\\ a-x & \frac{a}{2}<x<a,\end{cases}$$

5. Obtain a half – range cosine series for f(x) 2x – 1 for $0<x<1$.
Hence show that

$$\frac{p^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\infty$$

6. Find a Fourier sine series for

a) $f(x)=\begin{cases}1 & 0<x<\frac{1}{2}\\ 0 & \frac{1}{2}<x<1\end{cases}$

b) $f(x)=x(p-x)\quad in\quad 0<x<p$

7) Expand f(x) =1 –x^2, -1 < x < 1 in a fourier series. (N 2001)

8) Obtain the Fourier series for $f(x)=e^{-x}\quad in\quad (0,2p)$ (N 2000)

9) Obtain the Fourier series for $f(x)=x^2$ in $(-p,p)$ (N 2001)

10) Obtain the Fourier series for $f(x)=e^{-ax}\quad in\quad (-p,n)$

and hence deduce that $\cos ehx=\frac{2}{p}\sum\frac{(-1)^n}{n^2+1}$ (A 2001)

11) Prove that in 0 < x < l

$$x=\frac{1}{2}-\frac{4l}{p^2}\left(\cos\frac{px}{l}+\frac{1}{3^2}\cos\frac{3px}{l}+\frac{1}{5^2}\cos\frac{5px}{l}+\ldots\right)$$

and deduce that

Answers

$(i)\ \sum\frac{1}{(2n-1)^4}=\frac{p^4}{96}\quad (ii)\ \sum\frac{1}{n^4}=\frac{p^4}{90}$

1) $f(x)=\frac{p}{2}-\sum\frac{2}{n^2p}\left[(-1)^n-1\right]\cos nx$

3) $f(x)=2p\sum\frac{n}{1+n^2p^2}\left[1-(-1)^n\right]\sin px$

4) $f(x)=\frac{a}{4}-\frac{8}{p^2}\left[\frac{1}{2^2}\cos\frac{2px}{a}+\frac{1}{6^2}\cos\frac{6px}{a}+\frac{1}{10^2}\cos\frac{10px}{a}+\ldots\right]$

6) a) $f(x)=\sum\frac{2}{np}(1-\cos\frac{np}{2})\sin px$

b) $f(x)=\frac{8}{p}\sum\frac{\sin(2n-1)x}{(2n-1)^3}$

9) $x^2=\frac{p^2}{3}-\sum_1^\infty\frac{4}{n^2}\cos np\cos nx$

10) $e^{-ax}=\frac{\sin ap}{ap}+\sum_1^\infty\frac{2a\sinh ap}{p(a^2+n^2)}(-1)^n\cos nx$

$$+\sum_1^\infty\frac{2a\sinh ap}{p(a^2+n^2)}(-1)^n\sin nx$$

EXERCISE

A. Define Half range a) cosine b) sine series

1. Find the cosine and sine series for f(x) = x in 0 ≤ x ≤ π and hence show that $\frac{p^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

2. Obtain the Fourier series for the periodic function f(x) defined by $f(x)=\begin{cases}1-x & for \quad -p<x<0\\ 1+x & for \quad 0<x<p\end{cases}$

and hence show that $\frac{p^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

3. Obtain the Fourier series of f(x) defined by

$$f(x)=\begin{cases}x+\frac{p}{2} & in \quad -p<x\le 0\\ \frac{p}{2}-x & in \quad 0\le x<p\end{cases}$$

4. Prove that the Fourier series expansion of x(π - x) defined in the interval (0, π) is $\frac{p^2}{6}-\left[\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\ldots\right]$

5. Obtain the Fourier series for the function

$$f(x)=\begin{cases}x^2 & for \quad 0<x<p\\ -x^2 & for \quad p\le x<p\end{cases}$$

6. $f(x)=\begin{cases}x & in \quad 0<x<\frac{p}{2}\\ p-x & in \quad \frac{p}{2}<x<p\end{cases}$

Show that (i) $f(x)=\frac{4}{p}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-+\ldots\right]$

(ii) $f(x)=\frac{p}{4}-\frac{2}{p}\left[\frac{\cos 2x}{1^2}+\frac{\cos 6x}{3^2}+\frac{\cos 10x}{5^2}+\ldots\right]$

7. If $f(x)=\begin{cases}px & in \quad 0\le x\le 1\\ p(2-x) & in \quad 1\le x\le 2\end{cases}$

in the interval (0, 2) find the Fourier series of f(x)

8. If $f(x)=\begin{cases}x & in \quad (0,l)\\ x-2l & in \quad (l,2l)\end{cases}$ find the Fourier series in (-π, π)

9. Find the half-range cosine series for sinx in (0, π)

10. Find the half – range sine series for f(x) = 2x – 1 in (0, 1)

11. Find the half – range cosine series for f(x) = x^2 on (0, π)

12. Find the half – range sine series for f(x) = x^2 in (0, π)

13. Find the Fourier series for f(x) = 1 + x + x^2 in (- π, π)

14. Express f(x) = 1 + x^2 as a Fourier series in (0, π)

15. Expand $f(x)=\begin{cases}x^2 & in \quad (-p,0)\\ 0 & in \quad (0,p)\end{cases}$ as a Fourier series in (- π, π)

$$\frac{p^2}{6}+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}\cos nx+\sum_{n=1}^{\infty}\left[\frac{(-1)^n p}{n}-\frac{[(-1)^n-1]^2}{p\,n^3}\right]\sin nx$$

16. If $f(x)=\begin{cases}0 & for \quad -p<x<0\\ \sin x & for \quad 0<x<p\end{cases}$

Prove that $\frac{1}{p}+\frac{\sin x}{2}-\frac{2}{p}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$ and hence show that

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-+\ldots=\frac{1}{4}(p-2)$$

17. If $f(x)=\begin{cases}0 & for \quad 0\le x<\frac{p}{2}\left(\frac{p}{2}\right),\ f\left(\frac{p}{2}\right)=\frac{p}{4}\\ \frac{p}{2} & for \quad \frac{p}{2}<x\le p\end{cases}$

prove that

$f(x)=\frac{p}{4}-\cos x+\frac{\cos 3x}{3}-\frac{\cos 5x}{5}+-\ldots$ and hence show that

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\ldots=\frac{p}{4}$$

18. $$f(x)=\begin{cases}1+\dfrac{2x}{p} & for\ -p\le x\le 0\\ 1-\dfrac{2x}{p} & for\quad 0\le x\le p\end{cases}$$

Prove that $f(x)=\dfrac{8}{p^2}\left[\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\ldots\right]$

19. For $f(x)=|x|$ in $(-p,p)$, prove that

$$f(x)=\frac{p}{2}-\frac{4}{p}\left(\cos x+\frac{\cos 3x}{9}+\frac{\cos 5x}{25}+\ldots\right)$$

20. For $f(x)=x\sin x$ in $(-p,p)$ find the Fourier series and hece deduce that $\dfrac{1}{1.3}-\dfrac{1}{3.5}+\dfrac{1}{5.7}-+\ldots=\dfrac{1}{4}(p-2)$

21. Prove that the Half – range fourier sine series for f(x) = π - x in (0, π) is $\displaystyle\sum_{1}^{\infty}\frac{2}{n}\sin nx$ [2 Marks]

22. Prove that the Half range sine series for f(x) = e^x in (0, 1) is

$$\sum\frac{2p\,n}{1+n^2p^2}[1-(-1)^n e]\sin np\,x$$

ANSWERS

1. $(i)\dfrac{p}{2}-\dfrac{4}{p}\left[\cos x+\dfrac{1}{3^2}\cos 3x+\dfrac{1}{5^2}\cos 5x+\ldots\right]$

   $(ii)2\left[\sin x-\dfrac{1}{2}\sin 2x+\dfrac{1}{3}\sin 3x-+\ldots\right]$

2. $\dfrac{p+2}{2}-\dfrac{4}{p}\left[\dfrac{1}{1^2}\cos x+\dfrac{1}{3^2}\cos 3x+\dfrac{1}{5^2}\cos 5x+\ldots\right]$

3. $\dfrac{4}{p}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\ldots\right)$

5. $-p^2-8\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\ldots\right)$

6. $\dfrac{2}{p}\left[\left(\dfrac{3p^2}{1}-\dfrac{4}{1^3}\right)\sin x+\dfrac{p^2\sin 2x}{2}+\left(\dfrac{3p^2}{3}-\dfrac{4}{3^3}\right)\sin x+\dfrac{p^2\sin 4x}{4}+\ldots\right]$

7. $\dfrac{p}{2}-\dfrac{4}{p}\left[\dfrac{\cos p\,x}{1^2}+\dfrac{\cos 3p\,x}{3^2}+\dfrac{\cos 5p\,x}{5^2}+\ldots\right]$

8. $\dfrac{2l}{p}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{np\,x}{l}$

9. $\dfrac{2}{p}-\dfrac{4}{p}\displaystyle\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$

10. $-\dfrac{2}{p}\left[\dfrac{\sin 2p\,x}{1}+\dfrac{\sin 2p\,x}{2}+\dfrac{\sin 6p\,x}{3}+\ldots\right]$

11. $\dfrac{p^2}{3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}4\cos nx}{n^2}$

12. $\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}2p}{n^2}+\frac{[(-1)^n-1]4}{p\,n^3}\right]\sin nx$

13. $1+\dfrac{p^2}{3}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 4}{n^2}\cos np+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2}{n}\sin nx$

14. $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{[(-1)^n-1]4}{n}-\frac{2[(-1)^n(1+p^2)-1]}{p\,n}\right\}\sin x$

20. $1-\dfrac{1}{2}\cos x-\dfrac{2}{1.3}\cos 2x+\dfrac{2}{2.4}\cos 3x-\dfrac{2}{3.5}\cos 4x-\ldots$

5.9 Finite Sine and Cosine Transforms

Definitions: If f(x) is a sectionally continuous function over some finite interval (0, *l*) of the variable x, then the finite Fourier Sine and Cosine Transforms of f(x) over (0, *l*)are defined by

$$F_s(n)=\int_0^l F_s\,dx \text{ where } n=1,2,3,\ldots$$

and $$F_s(n)=\int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx \text{ where } n=1,2,\ldots$$

In the interval (0, p) we have

$$F_s(n)=\int_0^p f(x)\cos nx\,dx \text{ where } n=1,2,3\ldots$$

and $$F_c(n)=\int_0^p f(x)\cos nx\,dx \quad n=1,2,\ldots$$

Using Fourier Sine and Cosine half-range series, the inverse transforms in the interval (0, l) are given by

$$f(x)=\frac{2}{l}\sum_{n=l}^{\infty}F_s(n)\,sin\left(\frac{np\,x}{l}\right)$$

and $$f(x)=\frac{1}{l}F_c(0)+\frac{2}{l}\sum_{n=l}^{\infty}F_c(n)\cos\left(\frac{np\,x}{l}\right)$$

where $$F_c(0)=\int_0^l f(x)dx$$

In the interval (0, p), the above result becomes

$$F(x)=\frac{2}{p}\sum_{n=l}^{\infty}F_s(n)\sin nx$$

$$F(x)=\frac{1}{p}F_c(0)+\frac{2}{p}\sum_{n=l}^{\infty}F_c(n)\cos nx$$

where $$F_c(0)=\int_0^p f(x)dx$$

<u>NOTE</u> : If the interval is not given in the problems, then we have to take the interval as (0, A).

WORKED EXAMPLES :

(1) Find the finite Fourier sine and cosine transforms of f(x) =1 in (0, p)

Solution : Given : f(x) =1, in (0, l) = (0, π) → (1)

We know $$F_s(n)=\int_0^l f(x)sin\left(\frac{np\,x}{l}\right)dx$$

$$=\int_0^p 1\sin nx\,dx \qquad \text{[using (1)]}$$

$$=\left[-\frac{Cosnx}{n}\right]_0^p$$

$$=\frac{1-cosnp}{n}$$

$$F_s(n)=\frac{1-(-1)^n}{n}$$

Also, $$F_c(0)=\int_0^l f(x)Cos\left(\frac{np\,x}{l}\right)dx$$

$$=\int_0^p 1C\text{os}\,nx\,dx \qquad \text{[using (1)]} \rightarrow (2)$$

$$=\left[\frac{Sinnx}{n}\right]_0^p$$

F(n) = 0 if n = 1, 2, 3, . . .

If n= 0 then $$F_c(0)=\int_0^p Cos0dx \qquad \text{[using (2)]}$$

$$=[x]_0^p=\pi$$

(2) Find the finite Fourier sine and Cosine transforms of f(x) = x in (0, l).

Solution : We know $$F_s(n)=\int_0^l F(x)sin\left(\frac{np\,x}{l}\right)dx$$

$$= \int_0^l x sin\left(\frac{np\,x}{l}\right)dx \text{ (using given data)}$$

Using Bernoullie's rule, we get

$$F_s(n) = \left[x\frac{\left(-\cos\frac{np\,x}{l}\right)}{\left(\frac{np}{l}\right)} - 1\frac{\left(-\sin\frac{np\,x}{l}\right)}{\left(\frac{np}{l}\right)^2}\right]_0^l$$

$$= \frac{-l}{np}\left[x\cos\frac{np\,x}{l}\right]_0^l + \frac{l^2}{n^2p^2}\left[-\sin\frac{np\,x}{l}\right]_0^l$$

$$= \frac{-l}{np}[l\cos np - 0] + \frac{l^2}{n^2p^2}[\sin np - \sin 0]$$

$$= \frac{-l^2}{np}[-1]^n$$

$$F_s(n) = \frac{l^2(-1)^{n+1}}{np} \text{ where n = 1, 2, 3, . . .}$$

Now $F_c(n) = \int_0^l f(x)\cos\frac{np\,x}{l}dx$

$$= \int_0^l x\cos\frac{np\,x}{l}dx \qquad \text{(using given data)}$$

Using Bernoullie's rule, we get

$$F_c(n) = \left[x\frac{\sin\frac{np\,x}{l}}{\frac{np}{l}} - 1\frac{\left(-\cos\frac{np\,x}{l}\right)}{\frac{n^2p^2}{l}}\right]_0^l$$

$$= \frac{l}{np}\left[x\sin\frac{np\,x}{l}\right]_0^l + \frac{l^2}{n^2p^2}\left[\cos\frac{np\,x}{l}\right]_0^l$$

$$= \frac{l}{np}(0-0) + \frac{l^2}{n^2p^2}(\cos np - \cos 0)$$

$$F_c(n) = \frac{l^2}{n^2p^2}[(-1)^n - 1] \text{ where n = 1, 2, 3, . . .}$$

If n = 0, $F_c(0) = \int_0^l x dx$

$$= \left[\frac{x^2}{2}\right]_0^l$$

$$F_c(0) = \frac{l^2}{2}$$

(3) For the function f(x) = x, find the finite Fourier sine and Cosine transforms in (0, p)

Solution Given : f(x) = x, (0, l) =(0, p) $\rightarrow$ (1)

We know $F_s(n) = \int_0^l f(x)\sin\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^p x\sin nx dx \qquad \text{[using (1)]}$$

Using Bernoullie's rule, we get

$$F_s(n) = \left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^p$$

$$= -\frac{1}{n}[x\cos nx]_0^p \;(\because \sin np = \sin 0 = 0)$$

$$= -\frac{1}{n}[p\cos np - 0]$$

$$F_s(n) = \frac{(-1)^{n+1}p}{n} \text{ where n = 1, 2, 3, . . .}$$

Also ,

$$F_c(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$$

$$= \int_0^l x\cos nx\,dx \qquad \text{[usaing (1)]}$$

$$= \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^p \qquad \text{(using Bernoullie's rule)}$$

$$F_c(n) = \frac{1}{n^2}[\cos nx]_0^p$$

$$F_c(n) = \frac{1}{n^2}(\cos np - \cos 0)$$

$$F_c(n) = \frac{1}{n^2}\{(-1)^n - 1\}$$

If n = 2, 4,6, . . . , $F_c(n) = 0$

If n = 1, 3, 5, ..., $F_c(n) = \frac{-2}{n^2}$

If n = 0, $F_c(0) = \int_0^p x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^p$$

$$F_c(0) = \frac{p^2}{2}$$

(4) Find the finite Fourier sine transform of f(x)= x^2 in (0, 2)

Solution Given : f(x) = x^2, (0, l) = (0, 2) $\rightarrow$ (1)

We know $F_s(n) = \int_0^l f(x) sin\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^2 x^2 sin\left(\frac{np\,x}{2}\right)dx \qquad \text{[using (1)]}$$

Using Bernoullie's rule,

$$F_s(n) = \left[x^2\frac{-\cos\frac{np\,x}{2}}{\frac{np}{2}} - (2x)\frac{-\sin\frac{np\,x}{2}}{\frac{n^2p^2}{4}} + (2)\frac{\cos\frac{np\,x}{2}}{\frac{n^3p^3}{8}}\right]_0^2$$

$$= \frac{-2}{np}\left[x^2\cos\left(\frac{np\,x}{2}\right)\right]_0^2 + \frac{16}{n^3p^3}\left[\cos\left(\frac{np\,x}{2}\right)\right]_0^2 \quad (\because \sin np = \sin 0 = 0)$$

$$= \frac{-2}{np}(4\cos np - 0) + \frac{16}{n^3p^3}(\cos np - \cos 0)$$

$$F_s(n) = \frac{8}{np}(-1)^{n+1} + \frac{16}{n^3p^3}[(-1)^n - 1]$$

(5) Find the finite Fourier sine and Cosine transforms of f(x) = π-x.

Solution : Since the range is not given we shall take the interval as (0, π)

We know $F_s(n) = \int_0^p (p - x)\sin nx\,dx (\because f(x) = p - x \text{ and } l = p)$

Using Bernoullie's rule,

$$F_s(n) = \left[(p - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^p$$

$$= \frac{-1}{n}[(p - x)\cos nx]_0^p \quad (\because \sin np = \sin 0 = 0)$$

$$= \frac{-1}{n}[(0 - p)]$$

$$F_s(n) = \frac{p}{n}$$

Also $F_c(n) = \int_0^p (p - x)\cos nx\,dx$

Using Bernoullie's rule,

$$F_s(n) = \left[(p - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^p$$

$$= \frac{-1}{n^2}[\cos nx]_0^p \quad (\because \sin np = \sin 0 = 0)$$

$$= \frac{-1}{n^2}[\cos np - \cos 0]$$

$$F_c(n) = \frac{-1}{n^2}\left[(-1)^n - 1\right] \text{ where n} \neq 0.$$

When n = 0, $F_c(0) = \int_0^p (p - x)\,dx$

$$= \left[p x - \frac{x^2}{2}\right]_0^p$$

$$= p^2 - \frac{p^2}{2}$$

$$F_c(0) = \frac{p^2}{2}$$

(6) Find the finite Fourier Sine and Cosine transforms of

f(x) = 2x - x^2

Solution : Since the range is not given, we shall take the interval as (0, π)

Given : $f(x) = 2x - x^2, (0, l) = (0, p)$ $\rightarrow$ (1)

We know $F_s(n) = \int_0^l f(x)\sin\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^p (2x - x^2)\sin nx\,dx \qquad \text{[using (1)]}$$

Using Bernoullie's rule,

$$F_s(n) = \left[(2x - x^2)\left(\frac{-\cos nx}{n}\right) - (2 - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^p$$

$$= \frac{-1}{n}\left[(2x - x^2)\cos nx\right]_0^p - \frac{2}{n^3}\left[\cos np\right]_0^p$$

$$= \frac{-1}{n}\left[(2p - p^2)\cos np - 0\right] \frac{-2}{n^3}[\cos np]_0^p$$

$$= \frac{-1}{n}\left[(2p - p^2)(-1)^n\right] \frac{-2}{n^3}[(-1)^n - 1]$$

$$F_s(n) = \frac{(-1)^{n+1}(2p - p^2)}{n} \frac{-2}{n^3}[(-1)^n - 1] \text{ where n = 1, 2, 3, . .}$$

Also $F_c(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$

$$= \int_0 (2x - x^2)\cos nx\,dx$$

Using Bernoullie's rule,

$$F_c(n) = \left[(2x - x^2)\left(\frac{\sin nx}{n}\right) - (2 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^p$$

$$= \left[\frac{1}{n^2}(2 - 2x)\cos nx\right]_0^p$$

$$= \frac{2}{n^2}[(1 - x)\cos nx]_0^p$$

$$= \frac{2}{n^2}[(1 - p)\cos np - \cos 0]$$

where n ≠ 0

If n = 0, $F_c(0) = \int_0^p (2x - x^2)\,dx$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^p$$

$$F_c(0) = p^2 - \frac{p^3}{3}$$

7) Show that the finite Fourier sine transform of f(x) =x(π-x) is $\frac{4}{n^3}$ if n is odd and 0 if n is even.

Solution : Since the range is not given, we shall take the interval as (0, π)

Given : f(x) =x(π-x), (0, l) = (0, π) $\rightarrow$ (1)

We know $F_s(n) = \int_0^l f(x)\sin\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^l x(p - x)\sin nxdx \qquad \text{[using (1)]}$$

$$= \int_0^p (p\,x - x^2)\sin nxdx$$

Using Bernoullie's rule,

$$F_s(n) = \left[(p\,x - x^2)\left(\frac{-\cos nx}{n}\right) - (p - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^p$$

$$= \frac{-1}{n}\left[(p\,x - x^2)\cos nx\right]_0^p - \frac{2}{n^3}[\cos nx]_0^p \; (\because \sin nx = \sin 0 = 0)$$

$$= \frac{-1}{n}[0 - 0] - \frac{2}{n^3}[\cos np - \cos 0]$$

$$= -\frac{2}{n^3}[(-1)^n - 1]$$

$$F_s(n) = \frac{2}{n^3}[1 - (-1)^n]$$

If n is odd, $F_s(n) = \frac{2}{n^3}[1 - (-1)]$

$$F_s(n) = \frac{4}{n^3}$$

If n is even, $F_s(n) = \frac{2}{n^3}[1 - 1] = 0$

Thus, $F_s(n) = \frac{4}{n^3}$ if n is odd and 0 if n is even.

(8) Show that the finite Fourier Cosine transform of

$$f(x) = \left(1 - \frac{x}{p}\right)^2 is \begin{cases} \frac{2}{p\,n^2}, \; if \;\; n = 1,2,3,\ldots \\ \frac{p}{3}, \qquad if \quad n = 0. \end{cases}$$

Solution : We shall take the interval as (0, π)

Given : $f(x) = \left(1 - \frac{x}{p}\right)^2, (0, l) = (0, p)$ $\rightarrow$ (1)

We know $F_s(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^p \left(1 - \frac{x}{p}\right)^2 \cos nxdx \qquad \text{[using (1)]}$$

Using Bernoullie's rule,

$$F_c(n) = \left[\left(1 - \frac{x}{p}\right)^2\left(\frac{\sin nx}{n}\right) - 2\left(1 - \frac{x}{p}\right)\left(\frac{-1}{p}\right)\left(\frac{\cos nx}{n^2}\right) + 2\left(\frac{-1}{p^2}\right)\left(\frac{-\sin nx}{n^3}\right)\right]_0^p$$

$$= \frac{-2}{p\,n^2}\left[\left(1 - \frac{x}{p}\right)\cos nx\right]_0^p \; (\because \sin np = \sin 0 = 0)$$

$$= \frac{-2}{p\,n^2}[0 - \cos 0]$$

$$F_c(n) = \frac{2}{p\,n^2} \quad for \quad n \neq 0$$

If n = 0, $F_c(0) = \int_0^p \left(1 - \frac{x}{p}\right)^2 dx$

$$= \left[\frac{\left(1-\frac{x}{p}\right)}{\frac{-3}{p}} \right]_0^p$$

$$= \frac{-p}{3}\left[\left(1-\frac{x}{p}\right)^3\right]_0^p$$

$$= \frac{-p}{3}[0-1]$$

$$F_c(0) = \frac{p}{3}$$

(9) Find the Fourier Cosine transform of f(x) defined by

$$f(x) = \begin{cases} 1, & 0 < x < \frac{p}{2} \\ -1, & \frac{p}{2} < x < p \end{cases}$$

Solution : Given (0, l) = (0, π)

We know $F_c(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$

$$= \int_0^p f(x)\cos nx dx$$

$$= \int_0^{p/2} f(x)\cos nx + \int_{p/2}^p f(x)\cos nx dx$$

$$= \int_0^{p/2} 1.\cos nx dx + \int_{p/2}^p (-1)\cos nx dx \qquad \rightarrow \qquad (1)$$

(using given data)

$$= \left[\frac{\sin nx}{n}\right]_0^{p/2} - \left[\frac{\sin nx}{n}\right]_{p/2}^p$$

$$= \frac{1}{n}\left[\sin n\left(\frac{p}{2}\right) - 0\right] - \frac{1}{n}\left[0 - \sin\frac{np}{2}\right]$$

$$F_c(n) = \frac{2}{n}\sin\frac{np}{2} \qquad \text{for } n \neq 0 \qquad \rightarrow \qquad (2)$$

When n = 0, $F_c(0) = \int_0^{p/2} 1.dx + \int_{p/2}^p (-1)\,dx$ [using (1)]

$$= [x]_0^{p/2} - [x]_{p/2}^p$$

$$= \left[\frac{p}{2} - 0\right] - \left[p - \frac{p}{2}\right]$$

$$F_c(0) = 0 \qquad \rightarrow \qquad (3)$$

Thus,

$$F_c(n) = \begin{cases} 0, & \text{for} \quad n = 0, 2, 4, 6, \ldots \\ (-1)^{(n-1)/2}\frac{2}{n}, & \text{for} \quad n = 1, 3, 5, \ldots \end{cases}$$

[Using (2) & (3)]

(10) Find the finite Fourier Cosine transform of the function

$$f(x) = \begin{cases} 1, & \text{for} \quad 0 < x \le p/2 \\ 0, & \text{for} \quad p/2 < x < p \end{cases}$$

Solution : Given : (0, l) = (0, π)

We know

$$F_c(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$$

$$= \int_0^p f(x)\cos nx dx$$

$$= \int_0^{p/2} f(x)\cos nx + \int_{p/2}^{p} f(x)\cos nx\,dx$$

$$= \int_0^{p/2} 1.\cos nx\,dx + \int_{p/2}^{p} 0\cos nx\,dx \qquad \text{(using given data)}$$

$$= \int_0^{p/2} \cos nx\,dx \qquad \rightarrow \qquad (1)$$

$$= \left[\frac{\sin nx}{n}\right]_0^{p/2}$$

$$= \frac{1}{n}\left[\sin\frac{np}{2} - \sin 0\right]$$

$$= \frac{1}{n}\left[\sin\frac{np}{2}\right]$$

$$F_c(n) = \begin{cases} 0, \text{if } n \text{ is even} \\ (-1)^{(n-1)/2}\dfrac{1}{n}, \text{if } n \text{ is odd} \end{cases}$$

If n = 0 $F_c(0) = \int_0^{p/2} \cos 0\,dx$ [using (1)]

$$= \int_0^{p/2} 1.dx$$

$$F_c(0) = \frac{p}{2}$$

Thus,

$$F_c(n) = \begin{cases} p/2, & \text{for} \quad n = 0 \\ 0, & n = 2,4,6,\ldots \\ \dfrac{1}{n}(-1)^{(n-1)/2}, & \text{for} \quad n = 1,3,5,\ldots \end{cases}$$

(11) Find the finite Fourier Cosine and Sine transforms of the function $f(x) = e^{ax}$ in $(0, l)$.

Solution : We know $F_c(n) = \int_0^l f(x)\cos\left(\frac{np\,x}{l}\right)dx$

$$F_c(n) = \int_0^l e^{ax}\cos\left(\frac{np\,x}{l}\right)dx \qquad \rightarrow \qquad (1)$$

Using $\int_0^l e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$ we get

$$F_c(n) = \left[\frac{e^{ax}\left\{a\cos\left(\frac{np\,x}{l}\right) + \frac{np}{l}\sin\left(\frac{np\,x}{l}\right)\right\}}{a^2 + \frac{n^2p^2}{l^2}}\right]_0^l$$

$$= \frac{l^2a}{l^2a^2+n^2p^2}\left[e^{ax}\cos\frac{np\,x}{l}\right]_0^l + \frac{npl}{l^2a^2+n^2p^2}\left[e^{ax}\sin\left(\frac{np\,x}{l}\right)\right]_0^l$$

$$= \frac{l^2a}{l^2a^2+n^2p^2}\left[e^{al}\cos^{np} - 1\right] + \frac{np\,l}{l^2a^2+n^2p^2}\left[e^{al}\sin np - 0\right]$$

$$F_c(n) = \frac{l^2a}{l^2a^2+n^2p^2}[(-1)^n e^{al} - 1] \quad \text{where n = 1,2,3, . . .}$$

when n = 0, we get

$$F_c(n) = \int_0^l e^{ax}dx \qquad \text{[using (1)]}$$

$$= \left[\frac{e^{ax}}{a}\right]_0^l$$

$$F_c(0) = \frac{e^{al}-1}{a}$$

Also, $F_s(n) = \int_0^l f(x)\sin\left(\frac{np\,x}{l}\right)dx$

$$F_s(n)=\int_0^l e^{ax}\sin\left(\frac{np\,x}{l}\right)dx$$

Using $\int_0^l e^{ax}\sin bx dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}$, we get

$$F_c(n)=\left[e^{ax}\left\{\frac{a\sin\left(\frac{np\,x}{l}\right)-\frac{np}{l}\cos\left(\frac{np\,x}{l}\right)}{a^2+\frac{n^2p^2}{l^2}}\right\}\right]_0^l$$

$$=\frac{l^2a}{l^2a^2+n^2p^2}\left[e^{ax}\sin\left(\frac{np\,x}{l}\right)\right]_0^l-\frac{np\,l}{l^2a^2+n^2p^2}\left[e^{ax}\cos\left(\frac{np\,x}{l}\right)\right]_0^l$$

$$=\frac{np\,l}{l^2a^2+n^2p^2}\left[e^{al}\cos np-1\right]$$

$F_s(n)=\frac{np\,l}{l^2a^2+n^2p^2}[1-(-1)^n e^{al}]$ where n = 1,2,3, . . .

(12) Find f(x) in (0, π) given that the finite Fourier Cosine transform is $F_c(n)=\frac{\cos(2np/3)}{(2n+1)^2}$

Solution : In the interval $l=\pi$, we know

$$f(x)=\frac{1}{l}f_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}f_c(n)\cos\left(\frac{np\,x}{l}\right)$$

Here $l=\pi$

$$f(x)=\frac{1}{p}f(0)+\frac{2}{p}\sum_{n=1}^{\infty}f_c(n)\cos nx \qquad \rightarrow \qquad (1)$$

Given : $f_c(n)=\frac{\cos(2np/3)}{(2n+1)^2}$

$\therefore f_c(0)=1$

Using these in (1), we get

$$f(x)=\frac{1}{p}+\frac{2}{p}\sum_{n=1}^{\infty}\frac{\cos(2np/3)}{(2n+1)^2}\cos nx$$

(13) Find f(x) in (0, π) given that the finite Fourier sine transform is $f_s(n)=\frac{1-\cos np}{n^2p^2}$

Solution : We know $f_s(n)=\frac{2}{l}\sum_{n=1}^{\infty}f_s(n)\sin\left(\frac{np\,x}{l}\right)$ in (0, l)

Here $l=\pi$

$$f(x)=\frac{2}{p}\sum_{n=1}^{\infty}F_s(n)\sin nx$$

$$f(x)=\frac{2}{p}\sum_{n=1}^{\infty}\frac{(1-\cos np)}{n^2p^2}\sin nx \qquad \text{(Using given data)}$$

$\because 1-\cos np=1-(-1)^n=0$
if n is even and 2 if n is odd.
Using this, we get

$$f(x)=\frac{2}{p}\sum_{n=1,3,5,\ldots}^{\infty}\frac{2}{n^2p^2}\sin nx$$

$$f(x)=\frac{4}{p^3}\left[\frac{\sin x}{(1)^2}+\frac{\sin 3x}{(3)^2}+\frac{\sin 5x}{(5)^2}+\ldots\right]$$

(14) Find f(x) in 0 < x < 4 Given that $F_c(0)=16$,

$f_c(n)=\frac{3}{n^2p^2}\left[(-1)^n-1\right]$ where n = 1, 2, 3, . . .

Solution : We know $f(x)=\frac{1}{l}f_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}f_c(n)\cos\left(\frac{np\,x}{l}\right)$ in $0<x<l$

Given : $l=4$

$$\therefore f(x)=\frac{1}{4}(16)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{3}{n^2p^2}[(-1)^n-1]\cos\left(\frac{np\,x}{4}\right)$$

(using given data)

$$f(x)=4+\frac{3}{2p^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{-2}{n^2}\cos\left(\frac{np\,x}{4}\right)$$

$$f(x)=4-\frac{3}{p^2}\left[\frac{1}{1^2}\cos\frac{p\,x}{4}+\frac{1}{3^2}\cos\frac{3p\,x}{4}+\frac{1}{5^2}\cos\frac{5p\,x}{4}+\ldots\right]$$

[NOTE : S_n [f(x)] denote the finite Fourier sine transform of f(x) and S_n^{-1} is its inverse.
Similarly C_n [f(x)] denote the finite Fourier Cosine transform of f(x) and C_n^{-1} is its inverse.]

(15) Show that $S_n^{-1}\left[\frac{1-\cos np}{n^3}\right]=\frac{1}{2}x(p-x)$

Solution : We shall prove that $\frac{1-\cos np}{n^3}=S_n\left[\frac{1}{2}x(p-x)\right]$

Here $l = p$

$$S_n\left[\frac{1}{2}x(p-x)\right]=\int_0^p\frac{1}{2}x(p-x)\sin nx\,dx$$

$$=\frac{1}{2}\int_0^p x(p-x)\sin nx\,dx$$

Using Bernoullie's rule,

$$S_n\left[\frac{1}{2}x(p-x)\right]$$

$$=\frac{1}{2}\left[(p-x^2)\left(\frac{-\cos nx}{n}\right)-(p-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^p$$

$$=\frac{1}{2}\left[\frac{-(p\,x-x^2)\cos nx}{n}-\frac{2\cos nx}{n^3}\right]_0^p$$

$$=\frac{1}{2}\left[\frac{-2\cos np}{n^3}+\frac{2}{n^3}\right]$$

$$S_n\left[\frac{1}{2}x(p-x)\right]=\frac{1-C\,\mathrm{os}\,np}{n^3}$$

$$\therefore S_n^{-1}\left[\frac{1-\cos np}{n^3}\right]=\frac{1}{2}x(p-x)$$

(16) Show that $C_n^{-1}\left[\frac{k\sin kp}{k^2-n^2}\right]=\cos k(p-x)$ where $k\neq n$.

Solution : To prove that $C_n[\cos k(p-x)]=\frac{k\sin kp}{k^2-n^2}$

Here $l = p$.
We know

$$[\cos k(p-x)]=\int_0^p\cos k(p-x)\cos nx\,dx$$

$$=\frac{1}{2}\int_0^p[\cos(kp-kx+nx)+\cos(kp-kx-nx)]dx$$

$$=\frac{1}{2}\int_0^p[\cos\{kp-(k-n)x\}]dx$$

$$+\frac{1}{2}\int_0^p[\cos\{kp-(k+n)x\}]dx$$

$$=\frac{1}{2}\left[\frac{\sin[kp-(k-n)x]}{-(k-n)}\right]_0^p+\frac{1}{2}\left[\frac{\sin[kp-(k+n)x]}{-(k+n)}\right]_0^p$$

$$=\frac{-1}{2(k-n)}[\sin np-\sin kp]+\frac{-1}{2(k+n)}[\sin(-np)-\sin kp]$$

$$=\frac{1}{2}\left[\frac{\sin kp}{k-n}+\frac{\sin kp}{k+n}\right]$$

$$C_n[\cos k(p-x)]=\frac{k\sin kp}{k^2-n^2}$$

$$C_n^{-1}\left[\frac{k\sin kp}{k^2-n^2}\right]=\cos k(p-x)$$

5.12 Finite Sine and Cosine Transforms of Derivatives.

In the interval (0, l) , we prove the following results.

(1) $F_s\left[f^{(r)}(x)\right] = \frac{-np}{l}F_c\left[f^{(r-1)}(x)\right]$

(2) $F_s\left[f^{(r)}(x)\right] = (-1)^n f^{(r-1)}(l) - f^{(r-1)}(0) + \frac{np}{l}F_s\left[f^{(r-1)}(x)\right]$

Proof : Fourier finite sine transform is given by

$$F_s\left[f^{(r)}(x)\right] = \int_0^l f^{(r)}(x)\sin\left(\frac{np\,x}{l}\right)dx$$

Using integration by parts,

$$F_s\left[f^{(r)}(x)\right] = \left[f^{(r-1)}(x)\sin\left(\frac{np\,x}{l}\right)\right]_0^l - \frac{np}{l}\int_0^l f^{(r-1)}(x)\cos\left(\frac{np\,x}{l}\right)dx$$

$$= \left[f^{(r-1)}(l)\sin np - f^{(r-1)}(0)\sin 0\right] - \frac{np}{l}\int_0^l f^{(r-1)}(x)\cos\left(\frac{np\,x}{l}\right)dx$$

$$F_s\left[f^{(r)}(x)\right] = -\frac{np}{l}F_c\left[f^{(r-1)}(x)\right] \quad \rightarrow \quad (1)$$

Also , $F_c\left[f^{(r)}(x)\right] = \int_0^l f^{(r)}(x)\cos\left(\frac{np\,x}{l}\right)dx$

Using integration by parts,

$$F_c\left[f^{(r)}(x)\right] = \left[f^{(r-1)}(x)\cos\left(\frac{np\,x}{l}\right)\right]_0^l + \frac{np}{l}\int_0^l f^{(r-1)}(x)\sin\left(\frac{np\,x}{l}\right)dx$$

$$= \left[f^{(r-1)}(l)\cos np - f^{(r-1)}(0)\cos 0\right] + \frac{np}{l}F_s\left[f^{(r-1)}(x)\right]$$

$$F_c\left[f^{(r)}(x)\right] = (-1)^n f^{(r-1)}(l) - f^{(r-1)}(0) + \frac{np}{l}F_s\left[f^{(r-1)}(x)\right] \rightarrow (2)$$

[**NOTE :** Using the above results (1) and (2), we obtain the following results in the interval (0, l)]

Using r = 1 in (1) and (2), we get

$$F_s\left[F^1(x)\right] = -\frac{np}{l}F_c\left[f(x)\right] \quad \rightarrow \quad (3)$$

$$F_c\left[f^1(x)\right] = [(-1)^n f(l) - f(0)] + \frac{np}{l}F_s\left[f(x)\right] \quad \rightarrow \quad (4)$$

Using r = 2 in (1) and (2), we get

$$F_s\left[f''(x)\right] = -\frac{np}{l}F_c\left[f'(x)\right]$$

$$F_s\left[f'(x)\right] = \frac{-np}{l}[(-1)^n f(l) - f(0)] - \frac{n^2p^2}{l^2}F_s[f(x)] \quad \rightarrow \quad (5)$$

[Using (4)]

Also, $F_c\left[f''(x)\right] = [(-1)^n f'(l) - f'(0)] + \frac{np}{l}F_s[f'(x)]$

$$F_c\left[f'''(x)\right] = (-1)^n f'(l) - f'(0) - \frac{n^2p^2}{l^2}F_c[f(x)] \rightarrow \quad (6)[\text{using } (3)]$$

In the interval (0, π), the above results becomes

$$F_s\left[f'(x)\right] = -nF_c[f(x)] \quad \rightarrow \quad (7)$$

$$F_c\left[f'(x)\right] = [(-1)^n F(p) - f(0)] + nF_s[f(x)] \quad \rightarrow \quad (8)$$

$$F_s\left[f''(x)\right] = -n[(-1)^n f(p) - f(0)] - n^2F_s[f(x)] \quad \rightarrow \quad (9)$$

$$F_s\left[f'''(x)\right] = [(-1)^n f'(p) - f'(0)] - n^2F_c[f(x)] \quad \rightarrow \quad (10)$$

WORKED EXAMPLES

(17) By employing the finite Fourier Cosine transform, solve the equation $Y'' + 3Y = e^{-x}$, $Y'(0) = Y'(p) = 0$.

Solution : Given : $Y'' + 3Y = e^{-x}$

Using finite Fourier Cosine transform, we get,

$$F_c[Y''] + 3F_c[Y] = F_c[e^{-x}] \quad \rightarrow \quad (1)$$

In the interval, (0, l), we have

$$F_c\left[f''(x)\right] = (-1)^n f''(l) - f'(0) - \frac{n^2p^2}{l^2}F_c[f(x)]$$

Here (0, l) = (0, π) and Y = f(x)

$$\therefore F_c\left[y''\right] = (-1)^n y'(p) - y'(0) - n^2F_c(y)$$

Given : $Y'(0) = Y'(p) = 0$

$$\therefore F_c\left[y''\right] = -n^2 F_c[y] \qquad \rightarrow \quad (2)$$

Also, $F_c[e^{-x}] = \int_0^p e^{-x} \cos nx dx$

$$= \left[\frac{e^{-x}(-1\cos nx + n\sin nx)}{1^2+n^2}\right]_0^p$$

$$= \frac{-1}{1^2+n^2}\left[e^{-x}\cos nx\right]_0^p$$

$$= \frac{-1}{1^2+n^2}\left[e^{-p}\cos np - 1\right]$$

$$F_c[e^{-x}] = \frac{-1}{1^2+n^2}\left[(-1)^n e^{-p} - 1\right] \text{ for } n \neq 0 \qquad \rightarrow \quad (3)$$

Using (2) and (3) in (1), we get

$$-n^2 F_c[y] + 3F_c[y] = \frac{-1}{1^2+n^2}\left[(-1)^n e^{-p} - 1\right]$$

$$(n^2-3)F_c[y] = \frac{1}{1^2+n^2}\left[(-1)^n e^{-p} - 1\right]$$

$$F_c[y] = \frac{+1[(-1)^n e^{-p} - 1]}{(1+n^2)(n^2-3)}$$

This is denoted by $f_c(n)$ for $n \neq 0$

$$\therefore f_c(0) = \frac{+(-1)^n e^{-p-1}}{(n^2+1)(n^2-3)} \qquad \rightarrow \quad (4)$$

Put n = 0 in (4)

$$\therefore \tilde{f}_c(0) = \frac{e^{-p}-1}{-3} = \frac{1-e^{-p}}{3} \qquad \rightarrow \quad (5)$$

Using inverse Fourier Cosine transform,

$$y = \frac{1}{p}\tilde{f}(0) + \frac{2}{p}\sum_{n=1}^{\infty}\tilde{f}(n)\cos nx$$

$$y = \frac{1}{3p}(1-e^{-p}) + \frac{2}{p}\sum_{n=1}^{\infty}\frac{(-1)^n e^{-p} - 1}{(n^2+1)(n^2-3)}\cos nx$$

(18) Employing the finite Fourier sine transform, solve the differential equation $2y'' + y = x^2$ in $0 \le x \le l$, given y(0) = y(l) = 0.

Solution : $2y'' + y = x^2$

Using finite Fourier sine transform, we get

$$2F_s[y''] + F_s[y] = F_s[x^2] \qquad \rightarrow \quad (1)$$

In (0, l), we get

$$F_s\left[f''(x)\right] = \frac{-np}{l}[(-1)^n f(l) - f(0)] - \frac{n^2p^2}{l^2}F_s[f(x)]$$

$$F_s\left[y''\right] = \frac{-np}{l}[(-1)^n y(l) - y(0)] - \frac{n^2p^2}{l^2}F_s[Y]\left[\because Y = f(x)\right]$$ Using

Y(0) = Y (l)= 0, we get

$$F_s\left[y''\right] = \frac{n^2p^2}{l^2}F_s[y] \qquad \rightarrow \quad (2)$$

Also, $F_s\left[x^2\right] = \int_0^l x^2 \sin\left(\frac{np\,x}{l}\right)dx$

Using Bernoullies rule,

$$F_s\left[x^2\right] = \left[x^2\frac{\left(-\cos\left(\frac{np\,x}{l}\right)\right)}{\frac{np}{l}} - 2x\frac{\left(-\sin\frac{np\,x}{l}\right)}{\frac{n^2p^2}{l^2}} + 2\frac{\left(\cos\frac{np\,x}{l}\right)}{\frac{n^3p^3}{l^3}}\right]_0^l$$

$$= -\frac{l}{np}\left[x^2\cos\frac{np\,x}{l}\right]_0^l + \frac{2l^3}{n^3p^3}\left[\cos\frac{np\,x}{l}\right]_0^l$$

$$= -\frac{l}{np}\left[l^2\cos np - 0\right] + \frac{2l^3}{n^3p^3}[\cos np - 1]$$

$$F_s[x^2] = \frac{(-1)^{n+1}l^3}{np} + \frac{2l^3}{n^3p^3}\left[(-1)^n - 1\right] \qquad \rightarrow \quad (3)$$

Using (2) and (3) in (1), we get,

$$-2\left[\frac{-n^2p^2}{l^2}F_s[y]\right]+F_s[y]=\frac{(-1)^{n+1}l^3}{np}+\frac{2l^3}{n^3p^3}\left[(-1)^n-1\right]$$

$$\left(\frac{l^2-2n^2p^2}{l^2}\right)F_s[y]=\frac{(-1)^{n+1}l^3}{np}+\frac{2l^3}{n^3p^3}\left[(-1)^n-1\right]$$

$$F_s[y]=\left[\frac{(-1)^{n+1}l^3}{np}+\frac{2l^3\left[(-1)^n-1\right]}{n^3p^3}\right]\left[\frac{l^2}{l^2-2n^2p^2}\right]$$

Using inverse finite Fourier sine transform, we get

$$y=\sum_{n=1}^{\infty}F_s(y)\sin\frac{np\,x}{l}$$

$$y=\sum_{n=1}^{\infty}\frac{2l^4}{l^2-2n^2p^2}\left[\frac{(-1)^{n+1}}{np}+\frac{2\left[(-1)^n-1\right]}{n^3p^3}\right]\sin\frac{np\,x}{l}$$

(19) Using the finite Fourier Sine transform, solve the differential equation $y''+ky=x^3$ in $0<x<p$ given that y(0) = y(p) = 0 and k is a non-integral constant.

Solution : Given : $y''+ky=x^3$

Using finite Fourier sine transform,

$$F_s[y'']+kF_s[y]=F_s[x^3] \qquad \rightarrow \quad (1)$$

In (0, p)

$$F_s[y'']=-n[(-1)^n y(p)-y(0)]-n^2F_s[y]$$

Using y(0) = y(p) = 0, we get

$$F_s[y'']=-n^2F_s[y] \qquad \rightarrow \quad (2)$$

Also, $F_s[x^3]=\int_0^p x^3\sin nx\,dx$

Using Bernoullie's rule,

$$F_s[x^3]=\left[x^3\left(\frac{-\cos nx}{n}\right)-3x^2\left(\frac{-\sin nx}{n^2}\right)+6x\left(\frac{\cos nx}{n^3}\right)-6\left(\frac{\sin nx}{n^4}\right)\right]_0^p$$

$$F_s[x^3]=\left[\frac{-x^3\cos nx}{n}+\frac{6x\cos nx}{n^3}\right]_0^p$$

$$=\left[\frac{-p^3\cos np}{n}+\frac{6p\cos np}{n^3}\right]$$

$$=p\cos np\left[\frac{6}{n^3}-\frac{p^2}{n}\right]$$

$$F_s[x^3]=(-1)^n p\left[\frac{6}{n^3}-\frac{p^2}{n}\right] \qquad \rightarrow \quad (3)$$

Using (2) and (3) in (1) we get

$$-n^2F_s[y]+kF_s[y]=(-1)^n p\,\frac{(6-n^2p^2)}{n^3}$$

$$\therefore F_s[y]=\frac{(-1)^n p\,(6-n^2p^2)}{n^3(k-n^2)} \qquad \rightarrow \quad (4)$$

Using inverse finite Fourier sine transform, we get

$$y=\frac{2}{p}\sum_{n=1}^{\infty}F_s(y)\sin nx$$

$$y=2\sum_{n=1}^{\infty}\frac{(-1)(6-p^2n^2)}{(k-n^2)n^3}\sin nx$$

EXERCISES

1. Find the finite Fourier Sine transforms of the following
   (a) x in (0, 1)
   (b) 2-x in (0, 2)
   (c) ax – x^2 in (0, a)
   (d) cos x
   (e) e^{-x}
2. Find the finite Fourier Cosine transforms of the following
   (a) x^2 in (0, 1)
   (b) x (3 - x) in (0, 3)
   (c) $1-\frac{x}{a}$ in (0, a)
3. Find the Fourier Cosine transform of the function

$$f(x)=\begin{cases} p-x & in \quad 0<x<p/2 \\ x & in \quad p/2<x<p \end{cases}$$

4. Find the Fourier Cosine transform of the function

$$f(x)=\begin{cases} 1 \ in \ 0<x<1 \\ 0 \ in \ 1<x<2 \end{cases}$$

5. Show that the Finite Fourier Sine transform of $\frac{x}{p}$ is $\frac{(-1)^{n+1}}{n}$
6. Show that the finite Fourier Sine transform of $f(x)=e^{ax}$ in $(0,p)$ is $\frac{n}{a^2+n^2}[1+(-1)^{n+1}e^{ax}]$
7. Find the finite Fourier Sine transform of
   (i) sin ax and (ii) cos ax.
8. Find the finite Fourier Cosine transform of sin ax.
9. Find f(x) in $(0,p)$ given.
   (a) $F_{s(n)}=\frac{1-\cos np}{n^3}$

   (b) $F_s(n)=\frac{p}{n}$
   (c) $F_c(n)=\frac{1-\cos np}{n^3}, n=1,2,3,....., F_c(0)=\frac{p^2}{2}$
   (d) $F_c(n)=\frac{p}{2n^2}, n=1,2,3,....., F_c(0)=\frac{p}{3}$
10. If k is a constant and 0 < x < *l*, then prove that

$$C_n^{-1}\left[\frac{kl^2}{k^2l^2+n^2p^2}\right]=\frac{\cosh k(l-k)}{\sinh al}$$

11. Solve the following differential equations.
   (a) $y''-2y=e^{-2x}, 0\le x\le p$, given $y'(0)=y'(p)=0$ using Fourier finite Cosine transform.
   (b) $y''-y=x\sin x$ in $0\le x\le p$ given $y(0)=y(p)=0$ using Fourier finite Sine transform.
   (c) $y''-y=e^x$ in $0\le x\le p$, given $y(0)=y(p)=0$ using Fourier finite Sine transform.
   (d) $2y'+y=\sin^2 x$ in $0\le x\le p$, given $y(0)=y(p)=0$ using Fourier finite Sine transform.
   (e) $y''+y=\sin\frac{x}{2}$, $0<x<p$ given $y'(0)=y'(p)=0$ using Fourier finite Cosine transform.

ANSWERS

1. (a) $\frac{(-1)^{n+1}}{np}$ (b) $\frac{4}{np}$
(c) $\frac{\{1-(-1)^n\}2a^3}{n^3p^3}$ (d) 0 for n = 1 and
$\frac{\{1-(-1)^{n+1}\}n}{n^2-1}$ for n = 2, 3, 4, …. (e) $\frac{n}{1+n^2}[1-(-1)^n e^{-p}]$

2. (a) $\frac{2(-1)^n}{n^2p^2}$ (b) $\frac{2(-1)^n}{n^2p^2}$

(c) $\frac{[(-1)^n-1]a}{n^2p^2}$

3. $\frac{1+(-1)^n}{n^2}-\frac{2}{n^2}\cos\frac{np}{2}$

4. $\frac{2}{np}\sin\frac{np}{2}$

7. (i) $F_s(n)=\begin{cases}0 & if\ \ n\neq a, a\ \ is\ an\ \text{int}\,eger\ and\ \ \ n=1,2,3,\ldots\\ \frac{p}{2} & in\ \ \ n=a, nisapositive\ \text{int}\,eger.\end{cases}$

(ii) $F_c(n)=\frac{n[1+(-1)^n\cos ap]}{n^2-a^2}$

8. $F_c(n)=\begin{cases}0 & if \quad n\neq a, \quad niseven\\ \frac{2a}{a^2-n^2} & if \quad n\neq a, \quad nisodd\end{cases}$

9. (a) $\frac{2}{p}\sum_{n=1}^{\infty}\left(\frac{1-\cos np}{n^3}\right)\sin np$ (b) $2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

(c) $\frac{p}{2}+\frac{2}{p}\sum_{n=1}^{\infty}\left(\frac{1-\cos np}{n^2}\right)\sin np$ (d) $\frac{1}{3}+\sum_{n=1}^{\infty}\frac{1}{n^2}\sin nx$

11. (a) $y=\frac{1-e^{-2x}}{4p}+\frac{4}{p}\sum_{n=1}^{\infty}\frac{(-1)^ne^{-2x}-1}{(n^2+2)(n^2+4)}\cos nx$

(b) $y=\frac{-p^2}{8}\sin x+\sum_{n=2,4,6,\ldots}^{\infty}\frac{4^n}{(n^2-1)^2}\sin nx$

(c) $y=\frac{2}{p}\sum_{n=1}^{\infty}\frac{[(-1)^ne^p-1]}{(1+n^2)^2}\sin nx$

(d) $y=2\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{1}{2n}+\frac{1}{n^2-4}\right)\sin nx$

(e) $y=\frac{2}{p}-\frac{4}{p}\sum_{n=1}^{\infty}\frac{1}{\left(4n^2-1\right)\left(n^2+1\right)}\cos nx$