4 DIFFERENTIAL CALCULUS

4.1 Limit of a real function of real variables

4.1 Definition : A function f(x) is said to tend to a limit l as x tends to 'a' if corresponding to an arbitary positive number $\in$. However small, there exists another positive number δ such that.

$|f(x) - l| < \in$ when $0 < |x - a| < \delta$

i.e. $l - \in < f(x) < l + \in$ when $a - \delta < x < a + \delta$

Notation: $\underset{x\to a}{Lt}\ f(x) = l$

Note : For a function to posses a limit at a point, it may or may not be defined at that point. For example :

$\underset{x\to 5}{Lt}\dfrac{x^3-125}{x-5} = 75$, but at x= 5 the function is not defined.

4.1.1 Algebra of Limits

If $\underset{x\to a}{Lt}\ f(x) = l$ and $\underset{x\to a}{Lt}\ g(x) = m$ then prove that

(i) $\underset{x\to a}{Lt}\ kf(x) = k.l$ where k is a constant

(ii) $\underset{x\to a}{Lt}\ [f(x) + g(x)] = l + m$

(iii) $\underset{x\to a}{Lt}\ [f(x) - g(x)] = l - m$

(iv) $\underset{x\to a}{Lt}\ \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$ provided $m \neq 0$

Proof : (i) Given $\underset{x\to a}{Lt}\ f(x) = l$

By definition $|f(x) - l| < \in_1$ When $0 < |x - a| < \delta$.

Choose $\in_1$ as $\dfrac{\in}{|K|}$

Then $|f(x) - l| < \dfrac{\in}{|k|}$ when $0 < |x - a| < \delta$

ie $|k||f(x) - l| < \in$ when $0 < |x-a| < \delta$

ie $|kf(x) - kl| < \in$ when $0 < |x - a| < \delta$

ie $\underset{x\to a}{Lt}\ kf(x) = k\,.\,l$

(ii)Given $\underset{x\to a}{Lt}\ f(x) = l \Rightarrow |f(x) - l| < \dfrac{\in}{2}$ when $0 < |x - a| < \delta_1$ (1)

And $\underset{x\to a}{Lt}\ g(x) = m \Rightarrow |g(x) - m| < \dfrac{\in}{2}$ when $0 < |x - a| < \delta_2$ (2)

Let δ be smaller of δ_1 and δ_2

Then $|(f(x) + g(x)) - (l + m)| = |(f(x) - l) + (g(x) - m)|$

$\leq |f(x) - l| + |g(x) - m|$

$< \dfrac{\in}{2} + \dfrac{\in}{2} = \in$ when $0 < |x - a| < \delta$

$\underset{x\to a}{Lt}\ [f(x)+(g)] = l + m.$

ie., limit of the sum of the functions = sum of the limits

(iii) Similarly, $\underset{x\to a}{Lt}\ [f(x) - g(x)] = l - m$ (proof as above)

ie., limit of Difference of two functions =Difference of their limits.

(iv) Given $\underset{x\to a}{Lt}\ f(x) = l \Rightarrow |f(x) - l| < \in_1$ when $0 < |x - a| < \delta_1$

$\underset{x\to a}{Lt}\ g(x) = m \Rightarrow |g(x) - m| < |x - a| < \delta_2$

Let δ be smaller of δ_1 and δ_2

Choose $\in_1 = \dfrac{\in}{2|m|}$, $\in_2 = \dfrac{\in}{2|l|}$

Now $|f(x).g(x) - m| = |f(x)g(x) - l\,g(x) + l\,g(x) - lm|$

$= |g(x)(f(x) - l) + l(g(x) - m)|$

$\leq |g(x)|\ |f(x) - l| + |l|\ |g(x) - m|$

$\leq |m|\ |f(x) - l| + |l|\ |g(x) - m|$

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