4. Calculus of Variations

Introduction : In differential calculus, we have studies the method of finding maxima and minima of a function of one or more variables. In this chapter we study the methods of findings the curves of maxima and minima of functions of variables curves.

4.1 Functionals

Let S be the set of all functions of a single variable x in an interval (x_1, x_2) **.** Then a function which assigns a unique real umber of each function in S, is called a **functional**.

In symbols, a functional F is a mapping from the set of all functions to the set of real numbers i.e., F : S $\rightarrow$ R

Examples : (i) $\int_a^b (2x + y + 3y'^2)dx$ is a functional

(ii) $\int_b^a \sqrt{a^2 + y'^2}\,dx$ is a functional

(iii) $\int_{x_1}^{x_2} f(x, y, y')dx$ is a functional

4.2 Total Differential and Variation

Let F(x, y, y') be a function involving the independent variable x, the dependent variable y and the derivative of the dependent variable w.r.t. the independent variable. Then for each value of x, there will be a value for y and a value for y'. If x is fixed and y is taken as an arbitrary function of x and y' the derivatives, we get a unique real value for F(x, y, y') for each function y.

$\therefore$ For a fixed x, F(x, y, y') will be a functional.

By Taylor's expansion for a function of two variables, we have

$$F(x, y+h, y'+k) = F(x, y, y') + \left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial y'}\right)F$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial y} + k\frac{\partial}{\partial y'}\right)^2 F + \dots$$

$$\therefore F(x, y+h, y'+k) - F(x, y, y') = \left(h\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial y'}\right)$$

$$+\frac{1}{2!}\left(\frac{h^2\partial^2 F}{\partial y^2} + 2hk\frac{\partial^2 F}{\partial y \partial y'} + k^2\frac{\partial^2 F}{\partial y'^2}\right)$$

This is the increment in F and is denoted by ΔF. If the second and higher degree terms in h and k are neglected, we get.

$$\Delta F = \left(h\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial y'}\right) \quad \dots (1)$$

This is called **total derivative** (differential) of F and is denoted by dF.

$$\therefore dF = h\frac{\partial F}{\partial y} + k\frac{\partial F}{\partial y'} \quad \dots (2)$$

If we replace F by y we get

$$dy = h\frac{\partial y}{\partial y} + k\frac{\partial y}{\partial y'} = h(1) + k(0) = h$$

If we replace F by y' we get

$$dy' = h\frac{\partial y'}{\partial y} + k\frac{\partial y'}{\partial y'} = h(0) + k(1) = k$$

$$\therefore dy' = k$$

$$\therefore dF = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial y'}dy' \quad \dots (3)$$

This is the total differential of F.

Choose $h = \varepsilon\eta(x)$ and $k = \varepsilon\eta'(x)$ where ε is an arbitrary small quantity, $\eta(x)$ is an arbitrary function and $\eta'(x)$ is the derivative of $\eta(x)$, then $F(x, y+\varepsilon\eta(x), y'+\varepsilon\eta'(x)) - F(x, y, y')$

$$= \varepsilon\eta(x)\frac{\partial F}{\partial y} + \varepsilon\eta'(x)\frac{\partial F}{\partial y'} \quad \ldots (4)$$

This is called the Variation of F and is denoted by δF

$$\therefore \delta F = \varepsilon\eta(x)\frac{\partial F}{\partial y} + \varepsilon\eta'(x)\frac{\partial F}{\partial y'}$$

If we replace F by y, we get

$$\delta y = \varepsilon\eta(x)\frac{\partial y}{\partial y} + \varepsilon\eta'(x)\frac{\partial y}{\partial y'} = \varepsilon\eta(x) \quad \ldots (5)$$

If we replace F by y', we get

$$\delta y' = \varepsilon\eta(x)\frac{\partial y'}{\partial y} + \varepsilon\eta'(x)\frac{\partial y'}{\partial y'} = \varepsilon\eta'(x)$$

$$\therefore \delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \ldots \quad \ldots (6)$$

Where $\delta y = \varepsilon\eta(x)$ . . . (7)

and $\delta y = \varepsilon\eta'(x)$ . . . (8)

4.3 Standard Properties

Theorem 1 : δ and $\frac{d}{dx}$ commute each other

i.e. $$\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right)$$

Proof : $\delta y = \varepsilon\eta(x)$

$$\therefore \frac{d}{dx}(\delta y) = \varepsilon\eta'(x) \quad \ldots (1)$$

As y changes to $y + \varepsilon\eta(x)$, $\frac{dy}{dx}$ changes to $\frac{dy}{dx} + \varepsilon\eta'(x)$

If x is treated as fixed, then $\frac{dy}{dx}$ becomes a functional

$$\therefore \delta\left(\frac{dy}{dx}\right) = \varepsilon n'(x)(\because \delta y' = \varepsilon\eta'(x)) \quad \ldots (2)$$

From (1) and (2) we get

$$\frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right)$$

Theorem 2 : δ and $\int$ commute each other

i.e. if $\int_{x_1}^{x_2} f(x, y, y')dx$ is a functional, then

$$\delta\int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} \delta f(x, y, y')dx$$

i.e. the variation of a functional associated with f(x, y, y') is equal to functional associated with the variation of f.

Proof :

$$\delta\int_{x_1}^{x_2} f(x, y, y')dx = \frac{\partial}{\partial y}[\int_{x_1}^{x_2} f(x, y, y')dx]\delta y + \frac{\partial}{\partial y'}[\int_{x_1}^{x_2} f(x, y, y')dx]\delta y'$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial}{\partial y} f(x, y, y')dx\right]\delta y + [\int_{x_1}^{x_2}\frac{\partial}{\partial y'} f(x, y, y')dx]\delta y'$$

$$= \int_{x_1}^{x_2}[\frac{\partial}{\partial y} f(x, y, y')dy + \frac{\partial}{\partial y'} f(x, y, y')\delta y']dx$$

$$= \int_{x_1}^{x_2} df(x, y, y')dx$$

$$\therefore \delta\int_{x_1}^{x_2} f(x, y, y')dx = \int_{x_1}^{x_2} \delta f(x, y, y')dx$$

Theorem 3 : The operator δ satisfies the sum, differences, product and quotient rules of differentiation.

i.e. $\partial(f \pm g) = \partial f \pm \partial g$

$\partial(fg) = f\partial g + g\partial f$

$\partial\left(\frac{f}{g}\right) = \frac{g\partial f - f\partial g}{g^2}$

$\partial(cf) = c\partial f$ where C is a constant

and f and g are functions of x, y, y'

Proof : Left as exercise as it is straight forward application of the definition of ∂.

4.4. Fundamental Problem of Calculus of Variations

4.4.1 Extremal Functional, Variational Problem

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

where y is a function of x defined over the interval $[x_1, x_2]$ such that $y(x_1) = y_1$ and $y(x_2) = y_2$.

Let S be the set of all of all functions defined over the interval $[x_1, x_2]$. $\therefore$ $y(x) \in S$

Now the problem of finding $y(x) \in S$ for which the integral 1 is a maximum or minimum (i.e. extremum) in comparison with the neighbouring functions namely $y + \varepsilon\eta(x)$ where ε is small quantity and η(x) is a function of x such that $\eta(x_1) = \eta(x_2) = 0$.

A necessary condition for the integral I to have an extremum is given by Euler's Equation.

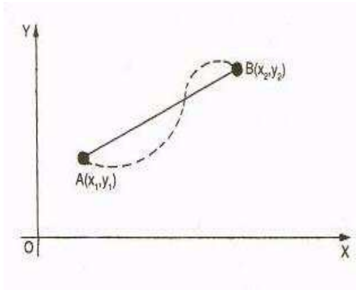
4.4.2 Euler's Equation

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')dx$

where $y(x_1) = y_1$ and $y(x_2) = y_2$, to have a maximum or a minimum is that $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Proof :

Let $$I = \int_{x_1}^{x_2} f(x, y, y')dx \quad \text{. . . (1)}$$



Let I be maximum or minimum along some curve y = y(x) passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

$\therefore$ A neighboring curve is given by

$$y = y(x) + \varepsilon\eta(x) \quad \text{. . . (2)}$$

Where ε is a small quantity and η(x) is a function of x such that $\eta(x_1) = \eta(x_2) = 0$

If ε = 0, then the neighbouring curve becomes

Y = y(x) which is the curve itself

This makes I an extremum

$$\therefore I = \int_{x_1}^{x_2} [f(x, y(x) + \varepsilon\eta(x); y'(x) + \varepsilon\eta'(x)]dx \quad \text{. . . (3)}$$

is an extremum when ε = 0.

By Leibnitz' rule for differentiation under the integral sign, we get

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} [\frac{\partial}{\partial\varepsilon} f(x, y(x) + \varepsilon\eta(x); y'(x) + \varepsilon\eta'(x)]dx$$

Let $Y(x) = y(x) + \varepsilon\eta(x)$

$\therefore Y'(x) = y'(x) + \varepsilon\eta'(x)$

$$\therefore \frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} f(x, Y, Y')dx = 0 \qquad \text{when } \in = 0$$

$$\frac{\partial}{\partial \varepsilon}[f(x), Y, Y'] = \frac{\partial f}{\partial Y}.\frac{\partial Y}{\partial \varepsilon} + \frac{\partial f}{\partial Y'}.\frac{\partial Y'}{\partial \varepsilon}$$

But $Y = y$ and $y' = y'$ when $\varepsilon = 0$

$$\therefore \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial y}.\frac{\partial y}{\partial Y'} = \frac{\partial f}{\partial Y'}$$

And $Y = y(x) + \varepsilon\eta(x) = \dfrac{\partial y}{\partial \varepsilon} = \eta(x)$

$$Y' = y'(x) + \varepsilon\eta'(x) = \frac{\partial y'}{\partial \varepsilon} = \eta'(x)$$

$$\therefore \left[\frac{dI}{d\varepsilon}\right]_{\varepsilon=0} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right]dx = 0 \qquad \text{. . . (4)}$$

This condition can be expressed in terms of variations as follows:

$$I = \int_{x_1}^{x_2} f(x, y, y')dx$$

$$\therefore \delta I = \delta \int_{x_1}^{x_2} f(x, y, y')dx$$

$$= \int_{x_1}^{x_2} \delta f(x, y, y')dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\partial y + \frac{\partial f}{\partial y'}\delta y'\right)dx$$

But $\partial y = \varepsilon\eta(x), \partial y' = \varepsilon\eta'(x)$

$$\therefore \partial I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}.\varepsilon\eta(x) + \frac{\partial f}{\partial y'}.\varepsilon\eta'(x)\right]dx$$

$$= \varepsilon \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right]dx$$

$= 0$ Using equation (4)

$\therefore$ the necessary condition for I to be extremum is ∂I = 0.

From (4) , $= \displaystyle\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right]dx = 0$

$$i.e. \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)dx = 0$$

$$i.e. \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)dx + \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx = 0$$

$$i.e. \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)dx + 0 - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx = 0$$

$$\because \eta(x_2) = \eta(x_1) = 0$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)dx = 0$$

Since $\eta(x)$ is arbitrary, $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

This is **Euler's equation** and is the condition for the extremum of the functional $\displaystyle\int_{x_1}^{x_2} f(x, y, y')dx$

4.4.3 Other forms of Euler's Equation

1. Since f(x, y, y') is a function of x, y, y', $\dfrac{\partial f}{\partial y'}$ is also a function of x, y, y'

$$\therefore \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y'}\right)\frac{dy'}{dx}$$

$$= \frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial^2 f}{\partial y \partial y'}.\frac{dy}{dx} + \frac{\partial^2 f}{\partial y'^2}.\frac{dy'}{dx}$$

$$=\frac{\partial^2 f}{\partial x\partial y'}+\frac{\partial^2 f}{\partial y\partial y'}.y'+\frac{\partial^2 f}{\partial y'^2}.y''$$

$\therefore$ Euler's equation $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$ becomes

$$\frac{\partial f}{\partial y}-\frac{\partial^2 f}{\partial x dy'}-\frac{\partial^2 f}{\partial y dy'}.y'-\frac{\partial^2 f}{\partial y'^2}.y''=0 \qquad \text{. . . (5)}$$

2. Since f is a function of x, y, y'

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.\frac{dy}{dx}+\frac{\partial f}{\partial y'}.\frac{dy'}{dx}$$

i.e. $\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.y'+\frac{\partial f}{\partial y'}.y''$ . . . (6)

and $\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial f}{\partial y'}.\frac{dy'}{dx}$

i.e. $\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{\partial f}{\partial y'}.y''$ . . . (7)

$\therefore$ (5) – (6)

$$\frac{df}{dx}-\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.y'+\frac{\partial f}{\partial y'}.y''-y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)-\frac{\partial f}{\partial y'}.y''$$

i.e., $\frac{d}{dx}\left[f-y'\frac{\partial f}{\partial y'}\right]=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}.y'-y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

$$=\frac{\partial f}{\partial x}+y'\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]$$

$$=\frac{\partial f}{\partial x}+y'(0)$$

$$\therefore \frac{d}{dx}\left[f-y'\frac{\partial f}{\partial y'}\right]=\frac{\partial f}{\partial x}$$

3. If f does not contain x explicitly, $\frac{\partial f}{\partial x}=0$

from (7) $\frac{d}{dx}\left[f-y'\frac{\partial f}{\partial y'}\right]=0$

$\therefore$ integrating w.r.t.x,

$f-y'\frac{\partial f}{\partial y'}=C$ where C is an arbitrary constant

$f-y'\frac{\partial f}{\partial y'}=C$ is a special form when f does not contain x explicitly.

4. If f does not contain y explicitly, $\frac{\partial f}{\partial y}=0$

From Eulers equation $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

i.e., $0-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

i.e., $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

Integrating w.r.t x, $\frac{\partial f}{\partial y'}=C$ where C is an arbitrary constant.

$\therefore$ When f doe not contain as explicitly, Euler's equation becomes

$$\frac{\partial f}{\partial y'}=C \qquad \text{. . . (9)}$$

5. If f does not contain both x and y explicitly, $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0$

In this case Euler's equation becomes $\frac{\partial^2 f}{\partial y'^2}=y''=0$

Using (4), we get $y''=0$

$\therefore y'' = 0$ is the condition if f does not contain both x and y explicitly.

Worked Examples

1 Obtain the Euler's equation to the extremal of

$$\int_{x_1}^{x_2} [y^2 + (xy')^2 + ye^x]dx \qquad \textbf{(N 04)}$$

Solution : Let $f = y^2 + (xy')^2 + ye^x$

$$\frac{\partial f}{\partial y} = 2y + e^x, \qquad \frac{\partial f}{\partial y'} = 2x^2 y'$$

Euler's equation $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 2y + e^x - \frac{d}{dx}\left(2x^2 y'\right) = 0$$

$$\Rightarrow 2y + e^x - \{2x^2 y'' + y'4x\} = 0$$

$$\Rightarrow -2x^2 y'' - 4xy' + 2y + e^x = 0$$

$$\Rightarrow 2x^2 y'' + 4xy' - 2y = e^x$$

2. Obtain the Euler's equation for the extremal of the functional

$$\int_{x_1}^{x_2} [y^2 - yy' + (y')^2]dx \qquad \textbf{(A 2004)}$$

Solution : Let $f = y^2 - yy' + (y')^2$

$$\frac{\partial f}{\partial y} = 2y - y', \qquad \frac{\partial f}{\partial y'} = -y + 2y'$$

Euler's equation is $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 2y - y' - \frac{d}{dx}\left(-y + 2y'\right) = 0$$

$$\Rightarrow 2y - y' + y' - 2y'' = 0$$

$$\Rightarrow y'' - y' = 0$$

3. Obtain the Euler's equation for solving the extremal problem : $\int_{x_1}^{x_2} y'(1 + x^2 y')dx$ **(M 2002)**

Solution : Let $f = y' + x^2 (y')^2$

$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = 1 + 2x^2 y'$$

Euler's equation is $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}\left(1 + 2x^2 y'\right) = 0$$

$$\Rightarrow -0 - 2x^2 y'' - 4xy' = 0$$

$$\Rightarrow \quad xy'' + 2y' = 0$$

4. Show that the Euler's equation for the extremum of $\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)dx$ **reduce to** $y'' - y = e^x$

Solution : Let $f = y^2 + y'^2 + 2ye^x$

$$\frac{\partial f}{\partial y} = 2y + 2e^x \quad ; \quad \frac{\partial f}{\partial y'} = 2y'$$

Euler's equation is $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 2y + 2e^x - \frac{d}{dx}\left(2y'\right) = 0$$

$$\Rightarrow y + e^x - y'' = 0$$

$$\Rightarrow y'' - y = e^x$$

5. Show that Euler's equation for the extremum of

$$\int_{x_1}^{x_2}[x^2(y')^2+4y(x+y)]dx=0 \textbf{ is } x^2y''+2xy'-4y=x$$

Solution : Let f $= x^2(y')^2+4y(x+y)$

$$\frac{\partial f}{\partial y}=4x+8y \quad ; \qquad \frac{\partial f}{\partial y'}=x^2 2y'=2x^2y'$$

Euler's equation is $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

$$\Rightarrow 4x+8y-\frac{d}{dx}\left(2x^2y'\right)=0$$

$$\Rightarrow 4x+8y-2x^2y''-4xy'=0$$

$$\Rightarrow x^2y''+2xy'-4y=2x$$

6. Solve the variation problem :

$$\delta\int_1^2[x^2(y')^2+2y(x+y)]dx=0 \text{ given that } y(1)=y(2)=0$$

Solution :

Euler's equations is :

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \qquad \ldots (1)$$

$$f=x^2(y')^2+2y(x+y)$$

$$\therefore \frac{\partial f}{\partial y}=0+2x+4y=2x+4y$$

$$\frac{\partial f}{\partial y'}=x^2 2y'+0=2x^2y'$$

Substitute in (1)

$$2x+4y-\frac{d}{dx}(2x^2y')=0$$

$$2x+4y-2(x^2y''+y'2x)=0$$

$$2x+4y-2x^2y''-4xy'=0$$

$$x^2y''+2xy'-2y=x$$

This is a differential equation of second order.

To solve this equation, use the substitution x = e^z.

$$\therefore xy'=Dy \text{ and } x^2y''=D(D-1)y \quad \text{where } D=\frac{d}{dz}$$

$\therefore$ [D(D-1)+2D -2]y = e^z

i.e., $(D^2+D-2)y=0$

AE is $D^2+D-2=0$

(D + 2) (D - 1) = 0

$\therefore$ D = -2, D = 1

$\therefore$ CF is $C_1e^{-2z}+C_2e^z$

$$PI=\frac{1}{D^2+D-2}e^z$$

$$=\frac{1}{(D+2)(D-1)}e^z$$

$$=z\frac{1}{D+2}e^z=z.\frac{1}{1+2}e^z=\frac{ze^z}{3}$$

Complete solution is y = CF + PI

i.e. $y=C_1e^{-2z}+C_2e^z+\frac{ze^z}{3}$

i.e., $y=C_1x^{-2}+C_2x+\log x.\frac{x}{3}$

i.e. $y=\frac{C_1}{x^2}+C_2x+\frac{x\log x}{3} \qquad \ldots (2)$

But y(1) = y(2) = 0 (given)

$$\therefore 0=\frac{C_1}{1^2}+C_2+\frac{1\log_1}{3}\Rightarrow C_1+C_2+0=0$$

$$\Rightarrow C_1+C_2=0 \qquad \ldots (3)$$

and $\quad 0=\frac{C_1}{2^2}+C_2(2)+\frac{2\log 2}{3}$

$$\Rightarrow \frac{C_1}{4} + 2C_2 = \frac{-2}{3}\log 2 \qquad \ldots (4)$$

solve (3) and (4)

(3) $\Rightarrow C_2 = -C_1$

$$\therefore (4) \Rightarrow \frac{C_1}{4} - 2C_1 = \frac{-2}{3}\log 2$$

$$\Rightarrow \frac{-7}{4}C_1 = \frac{-2}{3}\log 2$$

$$\Rightarrow C_1 = \frac{8}{21}\log 2$$

$$\therefore C_2 = -C_1 \Rightarrow \frac{-8}{21}\log 2 = C_2$$

$$\therefore \text{ Solution is } \quad y = \frac{8\log 2}{21x^2} - \frac{8\log 2}{21}.x + \frac{x\log x}{3}$$

7. Solve the variational problem

$$\delta\int_1^1 x^2(y')^2 dx = 0$$ **, Given y(1) = 1, y(2) = 1**

Given

Solution :

Euler's Equation is :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots (1)$$

Here $f = x^2 (y')^2$

$$\therefore \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = x^2.2y' = 2x^2 y'$$

$$\therefore (1) \Rightarrow 0 - \frac{d}{dx}(2x^2 y') = 0$$

$$\Rightarrow -(2x^2.y'' + y'.4x) = 0$$

$$\Rightarrow x^2 y'' + 2xy' = 0$$

Put $x = e^z$ or $z = \log x$

$$\therefore xy' = Dy \text{ and } x^2 y'' = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

$\therefore$ [D(D-1) + 2D] y = 0

i.e., $(D^2 + D)y = 0$

AE is $D^2 + D = 0$

D (D + 1) = 0

$\therefore$ D = 0, D = -1

$\therefore$ CF is $C_1 e^{0z} + C_2 e^{-z}$

i.e. $C_1 + C_2 e^{-z}$

$$PI = \frac{1}{D^2 + D}(0) = 0$$

$\therefore$ Complete solution is y = CF + PI

i.e. $y = C_1 e^{-2z} + C_2 e^{-z}$

$$\text{i.e. } y = C_1 + \frac{C_2}{x} \qquad \ldots (2)$$

It is given that y (1) = y (2) = 1.

$$1 = C_1 + \frac{C_2}{x} \Rightarrow C_1 + C_2 = 1 \qquad \ldots (3)$$

$$1 = C_1 + \frac{C_2}{x} \Rightarrow 2C_1 + C_2 = 2 \qquad \ldots (4)$$

(4) – (3) $\Rightarrow C_1 = 1$

(3) $\Rightarrow C_2 = 0$

$\therefore$ (2) $\Rightarrow$ Complete solution is y = 1

8. Find the extremal of the functional : (M05)

$$\int_{x_1}^{x_2} [y^2 + (y')^2 + 2y\, sech\, x]dx$$

Solution : $f = y^2 + (y')^2 + 2y \sec hx$

$$\therefore \frac{\partial f}{\partial y} = 2y + 2\sec h$$

$$\frac{\partial f}{\partial y'} = 2y'$$

$\therefore$ Eulers equation $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow \quad 2y + 2\sec h\, x - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow \quad 2y + 2\sec h\, x - 2y'' = 0$$

$$\Rightarrow \quad y'' - y = \sec h\, x \qquad \ldots (1)$$

This is a second order differential equation :

AE is $D^2 - 1 = 0$

$\therefore$ D = 1 , -1

$\therefore$ CF is $C_1 e^x + C_2 e^{-x}$

Replace C_1 and C_2 by A and B respectively which are functions of x.

$\therefore$ $y = Ae^x + B e^{-x}$

$$\frac{dy}{dx} = Ae^x + A'e^x + B'e^{-x} - Be^{-x}$$

$$= (Ae^x - Be^{-x}) + (A'e^x + B'e^{-x})$$

Choose $A'e^x + B'e^{-x} = 0 \qquad \ldots (2)$

$$\therefore \frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\therefore \frac{d^2y}{dx^2} = Ae^x + A'e^x - B'e^{-x} + Be^{-x}$$

Substituting in the equation (1): we get,

$$(Ae^x + A'e^x + Be^{-x} - B'e^{-x}) - (Ae^x + Be^{-x}) = \sec hx$$

i.e., $\quad Ae^x - Be^{-x} = \sec hx \qquad \ldots (3)$

Solve for A' and B' from (2) and (3);

$$A' = \frac{\begin{vmatrix} 0 & e^{-x} \\ \sec h & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-\sec hx.e^{-x}}{-1-1}$$

$$= \frac{1}{2}\sec hx.e^{-x}$$

$$= \frac{1}{2}.\frac{2}{e^x + e^{-x}}.e^{-x} = \frac{e^{-x}}{e^x + e^{-x}}$$

$$A' = \frac{e^{-x}}{e^x + e^{-x}}$$

$$B' = -A'e^{2x} = \frac{1}{1 + e^{-2x}} = \frac{e^{2x}}{e^{2x} + 1}$$

Integrating these w.r.t.x, we get

$$A = \int \frac{e^{-x}}{e^x + e^{-x}}\,dx$$

$$= \int \frac{e^{-2x}}{1 + e^{-2x}}\,dx$$

$$A = -\frac{1}{2}\log(1 + e^{-2x}) + C_1$$

$$B = \int \frac{e^{2x}}{1 + e^{2x}}\,dx = \frac{1}{2}\log(e^{2x} + 1) + C_2$$

$$\therefore (2) \Rightarrow y = [-\frac{1}{2}\log(1 + e^{-2x}) + C_1]e^x$$

$$+[+\frac{1}{2}\log(e^{2x} + 1) + C_2]e^{-x}$$

Where C_1 and C_2 are constants which can be determined using the values of y at x_1 and x_2.

9. Find the extremal of the functional :

$\int_0^1 \sqrt{1+(y')^2}$ **dx given that y (0)= 1 and y(1) =2**

(M 2001)

Solution :

$$f=\sqrt{1+(y')^2}$$

$$\frac{\partial f}{\partial y}=0, \quad \frac{\partial f}{\partial y'}=\frac{1}{2\sqrt{1+(y')^2}}.2y'$$

$\therefore$ Euler's equation is $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

$$\Rightarrow 0-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)=0$$

$$=\frac{y'}{\sqrt{1+(y')^2}} \text{ = Constant say C}$$

$$\Rightarrow y'=c\sqrt{1+(y')^2}$$

Squaring

$$(y')^2=c^2[1+(y')^2]$$

i.e. $(y')^2[1-c^2]=c^2$

$$\therefore (y')^2=\frac{c^2}{1-c^2}$$

$$\therefore y'=\frac{c}{\sqrt{1-c^2}}$$

$$\therefore dy=\frac{c}{\sqrt{1-c^2}}dx$$

$\therefore$ integrating we get

$$y=\frac{c}{\sqrt{1-c^2}}\int dx \text{ + constant B.}$$

i.e. $y=\frac{cx}{\sqrt{1-c^2}}+B$

It is given that $y(0)=1$ and $y(1)=2$

$$\therefore 1=0+B \qquad \therefore B=1$$

and $2=\frac{c}{\sqrt{1-c^2}}+1 \therefore 1=\frac{c}{\sqrt{1-c^2}}$

$$\therefore 1-c^2=c^2$$

$$\Rightarrow 2c^2=1$$

$$c^2=\frac{1}{2};\ c=\frac{1}{\sqrt{2}}$$

$\therefore$ Complete solution :

$$y=\frac{x}{\sqrt{2}\sqrt{1-\frac{1}{2}}}+1$$

$$\Rightarrow y=x+1$$

10. Prove that the extremal of $\int_0^1 \frac{(y')^2}{x}dx$ **with y(0)= 0, y(2) = 1 is a parabola** **(A 2004)**

Solution : Given $f=\frac{(y')^2}{x}$

$$\frac{\partial f}{\partial y}=0 \qquad \frac{\partial f}{\partial y'}=\frac{2y'}{x}$$

Euler's equation is $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

$$0-\frac{d}{dx}\left(\frac{2y'}{x}\right)=0$$

$$\Rightarrow \frac{2y'}{x}=c_1$$

$$\Rightarrow 2dy = c_1 x dx$$

$$\Rightarrow 2y = c_1 \frac{x^2}{2} + c_2 \qquad \ldots (1)$$

At $x = 0,\quad y = 0 \quad \Rightarrow 0 = 0 + c_2 \quad \therefore c_2 = 0$

At $x = 2,\quad y = 1 \quad \Rightarrow 2 = c_1 2 + 0 \quad \therefore c_1 = 1$

$(1) \Rightarrow \quad 2y = \frac{x^2}{2} \quad \therefore x^2 = 4y$

This is a parabola.

11. Show that the curve passing through (1, 0) and (2, 1) with $\int_1^2 \sqrt{\frac{1+(y')^2}{x^2}}dx$ is a circle **(A 2004 , 06)**

Solution : Let $f = \frac{1}{x}\sqrt{1+(y')^2}$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{1}{x}\frac{2y'}{2\sqrt{1+(y')^2}} = \frac{y'}{\sqrt{1+(y')^2}}$$

The Euler's equation becomes

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$0 - \frac{d}{dx}\left(\frac{y'}{x\sqrt{1+(y')^2}}\right) = 0$$

$$\Rightarrow \frac{y'}{x\sqrt{1+(y')^2}} = c_1$$

$$\frac{y'}{\sqrt{1+(y')^2}} = c_1 x$$

Squaring both sides and cross-multiplying

$$(y')^2 = c_1^2 x^2 (1 + y')^2)$$

$$(y')^2[1 - c_1^2 x^2] = c_1^2 x^2$$

$$y' = \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}$$

$$dy = \frac{c_1 x dx}{\sqrt{1 - c_1^2 x^2}}$$

Integrating $\quad y = -\frac{1}{c_1}\sqrt{1 - c_1^2 x^2} + c_2 \qquad \ldots (1)$

when x = 1, y = 0 $\quad \Rightarrow 0 = -\frac{1}{c_1}\sqrt{1 - c_1^2} + c_2$

$$\Rightarrow 1 - c_1^2 = c_1^2 c_2^2 \Rightarrow c_1^2 = \frac{1}{1 + c_2^2}$$

when x = 2, y = 1 $\quad \Rightarrow 1 = -\frac{1}{c_1}\sqrt{1 - 4c_1^2} + c_2$

$$\Rightarrow (1 - c_2)c_1 = -\sqrt{1 - 4c_1^2}$$

$$\Rightarrow (1 - c_2)^2 c_1^2 = 1 - 4c_1^2$$

$$\Rightarrow (1 - c_2)^2 \frac{1}{1 + c_2^2} = 1 - \frac{4}{1 + c_2^2} = \frac{c_2^2 - 3}{1 + c_2^2}$$

$$\Rightarrow (1 - c_2)^2 = c_2^2 - 3$$

on solving $\quad \Rightarrow c_2 = 2 \quad \therefore c_1 = \frac{1}{\sqrt{5}}$

$$(1) \Rightarrow y = -\sqrt{5}\sqrt{1 - \frac{x^2}{5}} + 2$$

$$(y - 2)^2 = \frac{5(5 - x^2)}{5}$$

$$(y - 2)^2 = (5 - x^2) \Rightarrow x^2 + y^2 - 4y - 1 = 0$$

The extremal of the given function is a circle.

12. Find the extremal of the functional

$$I=\int_0^{\pi/2}(y^2-y'^2-2y\sin x)dx$$ **under the condition**

$y(0)=y(\pi/2)=0$

Solution : I is maximum or minimum if it satisfies Euler's equations

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \qquad \text{. . . (1)}$$

$\Rightarrow f=y^2-y'^2-2y\sin x$

$\frac{\partial f}{\partial y}=2y-2\sin x$; $\frac{\partial f}{\partial y'}=-2y'$

$(1)\Rightarrow \quad 2y-2\sin x-\frac{d}{dx}(-2y')=0$

$\Rightarrow y''+y=\sin x$

It is a second order differential equation with constant coefficient.

Auxilaliary equation is $m^2+1=0 \quad \Rightarrow m=\pm i$

$C.F=A\cos x+B\sin x$

$P.I.=\frac{1}{D^2+1}\sin x=-\frac{x}{2}\cos x$

$\therefore y=A\cos x+B\sin x-\frac{x}{2}\cos x$ gives the computer solution

$y(0)=A=0$ From data and

$y(\pi/2)=0+B-\frac{\pi}{4}(0)=0 \quad \therefore B=0$

Thus the extremal value of I is $y=-\frac{x}{2}\cos x$

13. Show that the general solution of Euler's equation for the Functional

$I=\int_a^b\frac{1}{y}\sqrt{1+(y')^2}\,dx$ is $(x-B)^2+y^2=R^2$ (A 06)

Solution : Given I function is independent of x. Thus the corresponding Euler's equation is

$f-y'\frac{\partial f}{\partial y'}=A,$ where A is a constant

$$f=\frac{\sqrt{1+(y')^2}}{y}\Rightarrow\frac{\partial f}{\partial y'}=\frac{1}{y}\frac{2y'}{2\sqrt{1+(y')^2}}=\frac{y'}{y\sqrt{1+(y')^2}}$$

Euler's equation becomes

$$\frac{\sqrt{1+(y')^2}}{y}-\frac{(y')^2}{y\sqrt{1+(y')^2}}=A$$

$$\frac{1+(y')^2-(y')^2}{y\sqrt{1+(y')^2}}=A$$

$1=Ay\sqrt{1+(y')^2}$

$1=A^2y^2(1+(y')^2)$

$\frac{1}{A^2y^2}=1+(y')^2\Rightarrow(y')^2=-1+\frac{1}{A^2y^2}$

$\therefore\frac{dy}{dx}=\frac{\sqrt{1-A^2y^2}}{Ay}$

$\therefore\frac{Ay}{\sqrt{1-A^2y^2}}dy=dx$

$-\frac{1}{A}(1-A^2y^2)^{1/2}=x-B$

$\frac{1}{A^2}(1-A^2y^2)=(x-B)^2$

$(x-B)^2=\frac{1}{A^2}-y^2$

$(x-B)^2 + y^2 = \frac{1}{A^2} = R^2$ which is a circle.

Exercise

I Form the Euler's equation for the following

1) $\int_{x_1}^{x_2} \frac{(y')^2}{y^2} dx$ 2) $\int_{x_1}^{x_2} \sqrt{y(1+(y')^2)} dx$

3) $\int_0^1 [(y')^2 + 12xy] dx$ 4) $\int_0^1 [y^2 - yy' + (y')^2] dx$

5) $\int_0^1 (y^2 + x^2 y') dx$ 6) $\int_0^2 [y^2 - (y')^2] dx$

7) $\int_1^2 [x^2 (y')^2 + 2y^2 + 2xy] dx$ 8) $\int_0^1 \sqrt{1+(y')^2} dx$

II Solve the following variation problems.

1. $\delta \int_1^1 [12xy + (y')^2 dx = 0$ given that y(0) = 3, y(1) = 6
2. $\delta \int_4^5 \sqrt{x(1+y'^2)} dx = 0$ given that y(4) = 0, y(5) = 4
3. $\delta \int_1^1 [x + y + (y')^2] dx = 0$ given that y = 1 when x = 0 and y = 2 when x = 1
4. $\delta \int_0^2 [y^2 - (y')^2] dx = 0$ given that y(0) = 0 and $y\left(\frac{\pi}{2}\right) = 2$

III. Find the function y which makes the following functional extremum

5. $\int_0^4 [xy' - (y')^2] dx$ given that y(0) = 0 and y(4) = 3.
6. $\int_1^4 \sqrt{x}.(y')^2 dx$ given that y(1) = 5, y(4) = 7
7. $\int_0^1 \left[\frac{1}{y}\sqrt{1+(y')^2}\right] dx$ given that y =1 when x = 0 and y = 2 when x = 1
8. $\int_0^4 \sqrt{y}\sqrt{1+(y')^2} dx$ given that y(0) = 1, y(4)= 5
9. $\int_1^2 \frac{1}{x}\sqrt{1+(y')^2} dx$ given that y = 0 when x = 1 and y = 1 when x = 2
10. $\int_{x_1}^{x_2} [1 + xy' + x(y')^2] dx$

IV 11. Find the curve which passes through P(0, 2) and
Q$\left(\frac{1}{2}, e + \frac{1}{e}\right)$ along which the integral
$\int_0^{1/2} [y^2 + 4(y')^2] dx$ is extremum.

12. Find the curve which makes $\int_0^{\pi} [(y')^2 + 2y \sin x] dx$ an extremum given that it passes through (0, 0) and (π, 0)

V Show that the extremal value of

a) $\int_{x_1}^{x_2} y^2 (y')^2 dx$ is $y = c_1 \sqrt{x - c_2}$

b) $\int_1^2 x^2(y')^2dx$, y(0) =1,y(2) =1 is $y = \frac{4}{3}\left(1 - \frac{1}{x^2}\right)$

c) $\int_0^1 [x + y + (y')^2]dx$, (0,1), (1,2) is $4y = x^2 + 3x + 4$

d) $\int_0^1 (y^2 + x^2 y')dx$, (0,0), (1,1) is y = x

e) $\int_0^1 [x^2(y')^2 + 2y^2 + 2xy]dx$ y(1) = y(2) = 0 is

$$y = \frac{1}{21}\left[8\left(\frac{1}{x^2} - x\right)\log 2 + 7x\log x\right]$$

f) $\int_0^1 \sqrt{1+(y')^2}dx$ (0, 1) (1, 2) is y = x +1

g) $\int_0^1 \frac{1}{x}\sqrt{1+(y')^2}dx$ (0, 0) (1, 1) is $x^2 + (y-1)^2 = 1$

h) $\int_0^1 [y^2 - yy' + (y')^2]dx$ (0, 1) (1,2) is y = c sinh (x + a)

i) $\int_0^{\pi/2} [(y')^2 - y^2 + 2xy]dx$ y(0)=0, $y\left(\frac{\pi}{2}\right) = 0$ is $y = x - \frac{1}{2}\pi inx$

j) $\int_0^{\pi/2} (y^2 - (y')^2 - 2y\sin x)dx,\quad y(0) = y(\pi/2)$ is $y = -\frac{1}{2}x\cos x$

Answers

I.

1. $y' = c_1 y$ 2) $y' = \sqrt{y - c_1}$ 3) $y'' = 6x$

4) $y' = \sqrt{y^2 + c^2}$ 5) y = x 6) $y'' + y = 0$

7) $x^2 y'' + 2xy' - 2y = x$ 8) $y' = c_1\sqrt{1+(y')^2}$

II. 1. $y = x^2 + 2x + 3$ 2. $y = 4\sqrt{x-4}$

3. $y = \frac{x^2}{4} + \frac{3x}{4} + 1$ 4. y = 2 sin x

III. 5. $y = \frac{x^2 - x}{4}$ 6. $y = 2\sqrt{x} + 3$

7. $x = 2 - \sqrt{5 - y^2}$ 8. $x = 2\sqrt{y-1}$

9. $y = 2 - \sqrt{5 - x^2}$ 10. $2y = a\log x - x + b$

IV. 11. y = 2 cosh 2x 12. y = -sin x

4.5. Standard Problems

4.5.1 Geodesics

Definition : Among all curves joining two points on a surface the curve which has minimum length is called a **geodesic.**

Example : Among all curves joining two points in a plane, the straight line joining the two points has the minimum length. Below we determine the geodesics on plane sphere and right circular cylinder.

Theorem 1 : Show that the shortest distance between two points in a plane is along the straight line joining them.

Solution :

Let y = y(x) be a curve joining two points P(x_1, y_1) and Q(x_2, y_2) in the xy –plane

The arc length PQ is given by :

$$I = \int_{x_1}^{x_2} \frac{ds}{dx}dx$$

$$= \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

i.e. $$I = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

We have to find the curve along which I is minimum.
Euler's equation is :

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \ldots (1)$$

$$f = \sqrt{1+(y')^2}$$

$$\frac{\partial f}{\partial y} = 0,\ \frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1+(y')^2}}.2y'$$

$$\therefore (1) \Rightarrow 0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0$$

$$\Rightarrow \sqrt{1+(y')^2}.y'' - y'.\frac{1}{2\sqrt{1+(y')^2}}.2y'y'' = 0$$

$$\Rightarrow y''\left[\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}\right] = 0$$

$$\Rightarrow y'' = 0$$

Integrating, $y' = a$ where a is an arbitrary constant.
Integrating again, we get
y = ax + b where b is an arbitrary constant.
This equation represents a straight line.
This passes through P(x_1, y_1) and Q(x_2, y_2)

$$\therefore y_1 = ax_1 + b \quad \text{and}$$

$$y_2 = ax_2 + b$$

Subtracting , we get $a = \dfrac{y_2 - y_1}{x_2 - x}$

$$\therefore y_1 = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 + b$$

$$\therefore b = y_1 - \frac{(y_2 - y_1)x_1}{x_2 - x_1}$$

$$b = \frac{x_2y_1 - x_1y_1 - x_1y_2 + y_1x_1}{x_2 - x_1} = \frac{x_2y_1 - x_1y_2}{x_2 - x_1}$$

$$y = ax + b$$

$$\Rightarrow y = \frac{y_2 - y_1}{x_2 - x_1}x + \frac{x_2y_1 - x_1y_2}{x_2 - x_1}$$

$$\Rightarrow (y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Theorem 2 : Prove that the shortest arc joining the two points on a sphere is the minor arc of the great circle through the points.

Solution : The equation of a sphere whose centre is the origin and radius = a is $x^2 + y^2 + z^2 = a^2$.

In spherical polar coordinates the equation of the sphere is r = a.

In spherical polar coordinates the elementary arc length is given by

$$\therefore ds = \sqrt{h_1^2 dr^2 + h_2^2 d\theta^2 + h_3^2 d\phi^2}$$

where $h_1 = 1, h_2 = r, h_3 = r\sin\theta$

$$\therefore ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin\theta d\phi^2}$$

$$r = a \Rightarrow dr = 0$$

$$\therefore ds = \sqrt{a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2}$$

i.e. $$ds = a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

Let P(a, θ_1, ϕ_1) and Q(a, θ_2, ϕ_2) be two points on the sphere r = a. Then the arc length between P and Q is :

$$s = \int_P^Q ds$$

i.e. $s = \int_{\theta_1}^{\theta_2} a\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$

For this arc length to be minimum, the functional :

$$I = \int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$

I is of the form :

$$I = \int_{x_1}^{x_2} \sqrt{1+\sin^2 x\,(y')^2}\,dx \text{ where x = θ, y = ϕ}$$

$$\therefore f = \sqrt{1+\sin^2 x\,(y')^2}$$

$$\frac{df}{dy} = 0$$

Euler's equation $\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$$

$$\Rightarrow \frac{df}{dy'} = c$$

$$\Rightarrow \frac{\partial}{\partial y'}\sqrt{1+\sin^2 x\,(y')^2} = c$$

$$\Rightarrow \frac{1}{2\sqrt{1+\sin^2 x\,(y')^2}}.\sin^2 x.2y' = c$$

$$\Rightarrow \sin^2 xy' = c\sqrt{1+\sin^2 x\,(y')^2}$$

$$\Rightarrow (\sin^2 xy')^2 = c^2[1+\sin^2 x\,(y')^2]$$

$$\Rightarrow \sin^4 x.(y')^2 = c^2[1+\sin^2 x\,(y')^2]$$

$$\Rightarrow (\sin^4 x.-c^2\sin^2 x)(y')^2 = c^2$$

$$\Rightarrow y' = \frac{c}{\sqrt{\sin^4 x - c^2\sin^2 x}}$$

$$\Rightarrow y' = \frac{c}{\sqrt{\sin^4 x(1-c^2\cos ec^2 x)}}$$

$$\Rightarrow y' = \frac{c\cos ec^2 x}{\sqrt{1-c^2(1+\cot^2 x)}}$$

Integrating we get

$$y = \int \frac{c\cos ec^2 x}{\sqrt{1-c^2(1+\cot^2 x)}}dx + \text{constant}$$

Put $C\cot x = t$

$$\therefore C\cos ec^2 x dx = -dt$$

$$\therefore y = \int \frac{-dt}{\sqrt{1-c^2-t^2}} + b$$

$$\Rightarrow y = \cos^{-1}\left(\frac{t}{\sqrt{1-c^2}}\right) + b$$

$$\Rightarrow y - b = \cos^{-1}\left(\frac{t}{\sqrt{1-c^2}}\right)$$

$$\Rightarrow \left(\frac{t}{\sqrt{1-c^2}}\right) = \cos(y-b)$$

$$c\cot x = \sqrt{1-c^2}\cos(y-b)$$

Replacing x by θ and y by ϕ we get

$$c\cot\theta = \sqrt{1-c^2}\cos(\phi - b)$$

$$\Rightarrow \frac{c\cos\theta}{\sin\theta} = \sqrt{1-c^2}(\cos\phi\cos b + \sin\phi\sin b)$$

Multiply both sides by asinθ, we get

$$ca\cos\theta = \sqrt{1-c^2}(\cos ba\sin\cos\phi + \sin ba\sin\theta\sin\phi)$$

In spherical polar coordinates

$$x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi,\ z = a\cos\theta$$

∴ the equation in Cartesian coordinates becomes

$$cz = \sqrt{1-c^2}(\cos b.x + \sin b.y)$$

$$\Rightarrow x\cos b + y\sin b - \frac{cz}{\sqrt{1-c^2}} = 0$$

which is the form Ax + By +Cz = 0 which represents a plane passing through the origin. The section of the sphere by the plane is the great circles which has two arcs between P and Q viz., the major arc and the minor arc. The minor arc has the minimum length.

∴ The manor arc has the shortest distance. This is the geodesic on the surface of a sphere.

Theorem 3 : Prove that the shortest distance between two points on a circular cylinder, when the points are not on a generator, is along the circular helix joining them.

Solution :

Let $x^2 + y^2 = a^2$ be the equation of a circular cylinder with z – axis as its axis.

Let ρ, ϕ, z be the cylindrical coordinates

$$\therefore ds = \sqrt{h_1^2(d\rho)^2 + h_2^2(d\phi)^2 + h_3^2(dz)^2}$$

$h_1 = 1, h_2 = \rho, h_3 = 1$ are the scale factors.

$$\therefore ds = \sqrt{(d\rho)^2 + \rho^2(d\phi)^2 + (dz)^2}$$

$\rho = a$ is the equation of the cylinder

$$\therefore d\rho = 0$$

$$\therefore ds = \sqrt{a^2(d\phi)^2 + (dz)^2}$$

∴ Let P(a, ϕ_1, z_1) and $\delta(a, \phi_2, z_2)$ be any two points.

∴ The arc length joining P and Q is given by

$$I = \int_{\phi_1}^{\phi_2} \frac{ds}{d\phi} d\phi$$

$$I = \int_{\phi_1}^{\phi_2} \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}\, d\phi$$, which has to be minimum.

$$I = \sqrt{a^2 + \left(\frac{dz}{d\phi}\right)^2}$$

∴ Eulers equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow y'' = 0$$

$$\Rightarrow \frac{d^2 z}{d\phi^2} = 0$$

Integrating, we get $\frac{dz}{d\phi} = a$

Integrating again, we get $z = a\phi + b$

Since the curve passes through P(a, ϕ_1, z_1) and $Q(a, \phi_2, z_2)$

We get $z_1 = a\phi_1 + b$

and $z_2 = a\phi_2 + b$

Solving these equations for a and b and substituting in $z = a\phi + b$, we get

$$z - z_1 = \frac{z_1 - z_2}{\phi_1 - \phi_2}(\phi - \phi_1)$$

z increase on the curve from z_1 to z_2 proportional to the increase of ϕ from ϕ_1 to ϕ_2.

∴ The curve is a **circular helix.**

Theorem 4 : Find the geodesics on a right circular cone.

Solution : In spherical polar coordinates the equation of a right circular cone of semivertical angle α with vertex at the origin and the axis along z – axis is θ = α

$$ds = \sqrt{h_1^2 dr^2 + h_2^2 d\theta^2 + h_3^2 d\phi^2}$$

$$h_1 = 1, h_2 = r, h_3 = r\sin\theta$$

$$\therefore ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

since θ = α, $d\theta = 0$

$$\therefore ds = \sqrt{dr^2 + r^2\sin^2\alpha d\phi^2}$$

$$= \sqrt{1+\sin^2\alpha . r^2\left(\frac{d\phi}{dr}\right)^2} .dr$$

If P(r_1, α, ϕ_1) Q(r_2, α, ϕ_2) are any two points on the cone, the arc length PQ is given by :

This has to be minimum :

If x = r, y = ϕ.

$$s = \int_P^Q ds = \int_{r_1}^{r_2} \sqrt{1+\sin^2\alpha . r^2\left(\frac{d\phi}{dr}\right)^2} dr$$

$$S = \int_{x_1}^{x_2} \sqrt{1+\sin^2\alpha . x^2 (y')^2} dx$$

$$f = \sqrt{1+\sin^2\alpha . x^2 (y')^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$\therefore$ Euler's Equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$\frac{\partial}{\partial y'}\sqrt{1+\sin^2\alpha . x^2 (y')^2} = c$$

i.e. $\frac{1}{2\sqrt{1+\sin^2\alpha . x^2 (y')^2}}\sin^2\alpha . x^2 . 2y' = c$

i.e., $\sin^2\alpha . x^2 y' = c\sqrt{1+\sin^2\alpha . x^2 (y')^2}$

Squaring

$$\sin^4\alpha . x^4 (y')^2 = c^2[1+\sin^2\alpha . x^2 (y')^2]$$

$$\Rightarrow y' = \frac{c}{\sin^2\alpha . x\sqrt{x^2 - c^2\cos ec^2\alpha}}$$

Integrating, we get

$$y = \frac{c}{\sin^2\alpha} . \int \frac{1}{x\sqrt{x^2 - (c\cos ec\alpha)^2}} dx + \text{constant}$$

$$y = \frac{c}{\sin^2\alpha} . \frac{1}{c\cos ec\alpha}\sec^{-1}\left(\frac{x}{c\cos ec\alpha}\right) + b$$

$$\Rightarrow y = \frac{10}{\sin\alpha}\sec^{-1}\left(\frac{x\sin\alpha}{c}\right) + b$$

i.e. $y - b = \frac{1}{\sin\alpha}\sec^{-1}\left(\frac{x\sin\alpha}{c}\right)$

Replacing x by r and y by ϕ, we get

$\frac{r\sin\alpha}{c} = \sec[(\phi - b)\sin\alpha]$ where c and b are constants.

Theorem 5 : Find the geodesics on the helicoid x = u cos v, u sin v, z = k v

Solution : Hint : The arc length of the helicoids is

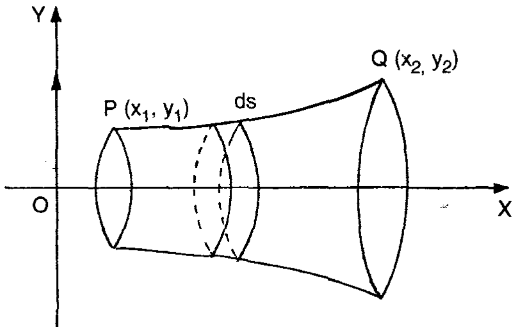
$$s = \int_{u_1}^{u_2} \sqrt{1+(u^2+k^2)\left(\frac{dv}{du}\right)^2} .du$$

4.5.2 Minimal Surface of Revolution

If a plane curve is rotated about a line in its plane, we get a surface of revolution.

In this section, we shall discuss about a curve which when rotated about a line gives a surface of revolution of minimum area.

Theorem : Find the curve passing through (x_1, y_1) and (x_2, y_2) which when rotated about the x-axis gives a minimum surface area.



Solution : Let P(x_1, y_1) and Q(x_2, y_2) be any two points on the curve Let ds be the arc length of PQ.

When the curve rotates about the x-axis, the elementary arc ds rotates through a distance 2πy round the x – axis.

∴ The elementary area = 2πy ds

$$= 2\pi y \frac{ds}{dx} dx$$

$$\therefore \text{ Total surface area } = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx} dx$$

i.e. $$S = 2\pi \int_{x_1}^{x_2} y\sqrt{1+(y')^2}\, dx$$

S has to be made minimum

Euler's equation is $\frac{df}{dy} - \frac{d}{dx}\left(\frac{df}{dy'}\right) = 0$

Here $f = y\sqrt{1+(y')^2}$

$$\frac{\partial f}{\partial y} = \sqrt{1+(y')^2}$$

$$\frac{\partial f}{\partial y'} = \frac{y}{2\sqrt{1+(y')^2}}.2y' = \frac{yy'}{\sqrt{1+(y')^2}}$$

∴ Euler's equation becomes

$$y\sqrt{1+(y')^2} - y'\frac{yy'}{\sqrt{1+(y')^2}} = c$$

i.e., $$\frac{y}{\sqrt{1+(y')^2}} = c$$

∴ $$c^2(1+(y')^2) = y^2$$

$$c^2(y')^2 = y^2 - c^2$$

$$y' = \frac{1}{c}\sqrt{y^2-c^2}$$

i.e. $$\frac{dy}{dx} = \frac{1}{c}\sqrt{y^2-c^2}$$

$$\therefore \frac{1}{c}x = \int \frac{dy}{\sqrt{y^2-c^2}}$$

Integrating we get

$$\therefore \frac{1}{c}x = \cosh^{-1}\left(\frac{y}{c}\right) + \text{constant}$$

i.e., $$\frac{1}{c}x = \cosh^{-1}\left(\frac{y}{c}\right) + \frac{a}{c}$$

$$\frac{x-a}{c} = \cosh^{-1}\frac{y}{c}$$

i.e., $y = c\cosh\left(\frac{x-a}{c}\right)$ where c and 'a' are constants which can be determined using the condition that the curve passes through (x_1, y_1) and (x_2, y_2). This equation represents the **catenary.**

4.5.3 Hanging Chain or Cable

When a heavy chain is suspended freely under gravity from two fixed points, then the wire take the shape of a curve called the **catenary**

Theorem 1 : A Chain hangs freely under gravity from two fixed points. Prove that the shape of the curve is a catenary.

Solution : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points from which the chain is suspended. If 'ds' is the length of an elementary arc of the chain, and is its density then ρ ds is the mass of the element of arc.

If x – axis is taken as the axis of reference, the potential energy is given by mgh.

i.e. $P.E. = (\rho ds)\, g\, y$

∴ Total $P.E. = \int_P^a (\rho ds)\, g\, y$

$$= \int_{x_1}^{x_2} \rho g y \frac{ds}{dx} dx$$

$$= \rho g \int_{x_1}^{x_2} y\sqrt{1+(y')^2}\, dx \quad \because \frac{ds}{dx} = \sqrt{1+(y')^2}$$

∴ We have to make the functional

$$I = \int_{x_1}^{x_2} y\sqrt{1+(y')^2}\, dx$$

$$f = y\sqrt{1+(y')^2}$$

$$\therefore \frac{\partial f}{\partial y} = \sqrt{1+(y')^2} \quad \text{and} \quad \frac{\partial f}{\partial y'} = y.\frac{1}{2\sqrt{1+(y')^2}} 2y'$$

∴ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$f - y'\frac{\partial f}{\partial y'} = C$ since f does not contain x explicity

$$\therefore y\sqrt{1+(y')^2} - y'\frac{y(y')^2}{\sqrt{1+(y')^2}} = c$$

i.e., $\frac{y[1+(y')^2] - y(y')^2}{\sqrt{1+(y')^2}} = c$

i.e. $\frac{y}{\sqrt{1+(y')^2}} = c$

$$\therefore y^2 = c^2[1+(y')^2] = c^2 + c^2(y')^2$$

$$\Rightarrow y' = \frac{\sqrt{y^2-c^2}}{c}$$

i.e. $\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$

$$\therefore \frac{dy}{\sqrt{y^2-c^2}} = cdx$$

Integrating, we get

$$Cosh^{-1}\left(\frac{y}{c}\right) = cx + b$$

$y = C\cosh(Cx+b)$ which is the equation of a catenary.

4.6 Brachistochrone Problem

This problem was proposed by the famous mathematician Bernoulli in the year 1696. This is a problem of quickest descent.

Brachistochrone Problem : To find the equation of the plane curve down which a particle acted upon by gravity would descend form one fixed point to another fixed point in the shortest possible time.

OR

To show that the path in which a particle in the absence of friction will slide form one fixed point to another fixed point in the shortest time under gravity is a **cycloid.**

Solution :

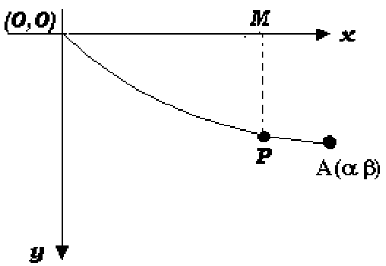
Let O(0, 0) be the point of starting and let A(α, B be the end point.

Let ox be the horizontal and oy downwards the vertical.

Since the particle moves under gravity without friction, the gain in the Kinetic Energy (K.E) in moving from 0 to any arbitrary point P(x, y) is equal to the loss of Potential Energy (P.E)

i.e., $\frac{1}{2}mv^2 = mgy$

But $v = \frac{ds}{dt}$



$\therefore \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mgy$

i.e., $\frac{ds}{dt} = \sqrt{2gy}$

Time taken by the particle to reach A(α, β) from O is

$$\int_0^T dt = \int_0^\alpha \frac{dt}{dx}dx$$

$$= \int_0^\alpha \frac{dt}{ds}.\frac{ds}{dx}dx$$

$$= \int_0^\alpha \frac{1}{\sqrt{2gy}}\sqrt{1+(y')^2}\,dx$$

We have to make the functional $\int_0^\alpha \sqrt{\frac{1+(y')^2}{y}}dx$ a minimum.

Here $f = \sqrt{\frac{1+(y')^2}{y}}$ which does not contain x explicitly.

$\therefore f - y'\frac{\partial f}{\partial y'} = c$, where C is a constant

Now, $\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}}.\frac{1}{2\sqrt{1+(y')^2}}2y'$

$$\therefore \sqrt{\frac{1+(y')^2}{y}} - y'\frac{y'}{\sqrt{y}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{1+(y')^2-(y')^2}{\sqrt{y}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow \frac{1}{\sqrt{y}\sqrt{1+(y')^2}} = c$$

$$\Rightarrow c^2y[1+(y')^2] = 1$$

$$\Rightarrow (y')^2 = \frac{1-c^2y}{c^2y}$$

i.e., $y' = \frac{\sqrt{1-c^2y}}{c\sqrt{y}}$

i.e., $\frac{dy}{dx} = \sqrt{\frac{\frac{1}{c^2} - y}{y}}$

i.e., $\frac{\sqrt{y}}{\sqrt{c^{1/2} y}} dy = dx$

Integrating, we get

$$\int \frac{\sqrt{y}}{\sqrt{\frac{1}{c^2} - y}} = dx + \text{constant}$$

Put $\frac{1}{c^2} = b^2$

$$\therefore \int dx = \int \frac{\sqrt{y}}{\sqrt{b^2 - y}} dy + c$$

Put $y = b^2 \sin^2 \frac{\theta}{2}$

$$\therefore dy = b^2 .2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} . \frac{1}{2} d\theta$$

$$\therefore \qquad x = \int \frac{b \sin\frac{\theta}{2}}{\sqrt{b^2 - b^2 \sin^2 \frac{\theta}{2}}} b^2 .2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} . \frac{1}{2} d\theta + c$$

i.e., $x = \int \frac{b \sin\frac{\theta}{2}}{\sqrt{b^2 \cos^2 \frac{\theta}{2}}} b^2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} d\theta + c$

i.e. $x = b^2 \int \sin^2 \frac{\theta}{2} d\theta + c$

$$= b^2 \int \frac{1}{2} (1 - \cos\theta) d\theta + c$$

$$x = \frac{b^2}{2}[\theta - \sin\theta] + c \quad \text{and} \quad y = b^2 \sin^2 \frac{\theta}{2}$$

i.e. $y = \frac{b^2}{2}(1 - \cos\theta)$

when θ = 0, y = 0 and x = 0

$\therefore$ (0,0) is a point on the curve $\qquad \therefore$ c = 0

$$x = \frac{b^2}{2}(\theta - \sin\theta), y = \frac{b^2}{2}(1 - \cos\theta)$$

By putting $\frac{b^2}{2}$ = a, we get

$$x = a(\theta - \sin\theta) \text{ and } y = a(1 - \cos\theta)$$

These are the parametric equations of a cycloid.

$\therefore$ The required curve is a **cycloid.**

4.7 ISOPERIMETRIC PROBLEMS :

Finding a closed curve of given perimeter and maximum area is called Isoperimetric problems

Usually an isoperimetric problem is as follows :

$$I = \int_{x_1}^{x_2} f(x_1, y_1, y')dx \qquad \text{. . . (1)}$$

under the conditions $y(x_1) = y_1$ and $y(x_2) = y_2$

subjected to the condition

$$\int_{x_1}^{x_2} h(x_1, y_1, y')dx = k \qquad \text{. .. (2)}$$

where k is a constant.

Solving any problem of this type is exactly similar to that of finding the extremal functional.

Worked Examples

1. Show that the sphere is the solid figure of revolution which for a surface area has maximum volume

Let $S = 2\pi\int yds = 2\pi\int y\sqrt{1+(y')^2}dx$ be the surface area

and

$$v = \pi\int_0^a y^2 dx$$

be volume of the given surface.

We have to maximise the function

$$H = f + \lambda g$$

$$= \pi y^2 + \lambda[2\pi y\sqrt{1+(y')^2}]$$

The Euler's equation

$H - y'\dfrac{dH}{dy'}$ = c becomes

$$\pi y^2 + \lambda[2\pi y\sqrt{1+(y')^2}] - y'\lambda[2\pi y\frac{\partial}{\partial y'}\sqrt{1+(y')^2}] = c$$

$$\pi y^2 + \lambda 2\pi y\sqrt{1+(y')^2} - y'\lambda[2\pi y\frac{2y'}{2\sqrt{1+(y')^2}}] = c$$

$$\Rightarrow \pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+(y')^2}} = c \qquad \text{. . . (1)}$$

When the curve crosses the x – axis y = 0 $\therefore$ c = 0

Thus $(1) \Rightarrow y + \dfrac{2\lambda}{\sqrt{1+(y')^2}} = 0$

$$\Rightarrow y = -\frac{2\lambda}{\sqrt{1+(y')^2}}$$

$$\Rightarrow y^2(1+(y')^2) = 4\lambda^2 \text{ on squaring}$$

$$\Rightarrow (y')^2 = \frac{4\lambda^2 - y^2}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$

$$\therefore \int\frac{y}{\sqrt{4\lambda^2 - y^2}}dy = \int dx$$

$$\Rightarrow \sqrt{4\lambda^2 - y^2} = x + k \qquad \text{. . . (2)}$$

when x = 0 , y = 0 $\Rightarrow k = \pm 2\lambda$

$$(2) \Rightarrow x \pm 2\lambda = \sqrt{4\lambda^2 - y^2}$$

$$\Rightarrow (x \pm 2\lambda)^2 + y^2 = (2\lambda)^2$$

This represent a circle. By revolution about the axis form a Sphere.

2. Find the extremal of the function

$$\int_0^1 [(y')^2 + x^2 + \lambda y]dx$$

under the condition y(0) = 0, y(1) = 0 and subjected to the constraint $\displaystyle\int_0^1 ydx = \frac{1}{6}$

Solution: Let $f = (y')^2 + x^2 + \lambda y$

$$\frac{\partial f}{\partial y} = \lambda, \qquad \frac{\partial f}{\partial y'} = 2y'$$

Euler's equation $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$ becomes

$$\lambda - \frac{d}{dx}(2y') = 0$$

$$\lambda - 2y'' = 0$$

$$2y'' = \lambda$$

$$\frac{d^2y}{dx^2} = \frac{\lambda}{2} \Rightarrow \frac{dy}{dx} = \frac{\lambda x}{2} + b$$

$$y = \frac{\lambda}{4}x^2 + bx + a \qquad \text{. . . (1)}$$

when $x = 0, y = 0 \Rightarrow 0 = a$ . . . (2)

$$x=1, y=0 \Rightarrow 0=\frac{\lambda}{4}+b \qquad \therefore b=-\frac{\lambda}{4} \qquad \ldots (3)$$

Also $\int_0^1 y\,dx=\frac{1}{6}$

$$\int_0^1 \left(\frac{\lambda}{4}x^2+bx+a\right)dx=\frac{1}{6}$$

$$\left.\frac{\lambda x^3}{12}+\frac{bx^2}{2}+ax\right]_0^1=\frac{1}{6}$$

$$\frac{\lambda}{12}+\frac{b}{2}+a=\frac{1}{6} \qquad \ldots (4)$$

$$\frac{\lambda+6b}{12}=\frac{1}{6} \qquad \because a=0$$

$$\lambda+6b=2$$

$$\lambda+6\left(-\frac{\lambda}{4}\right)=2 \quad \Rightarrow \lambda=-4 \quad b=1$$

$(1) \Rightarrow \quad y=-x^2+x$ is the required function.

3. Find the Extremal of the functional

$$I=\int_0^1 [x^2+\lambda y^2-(y')^2]dx$$

under the conditions (0, 0) (1, 0) and subject to the constraint $\int_0^1 y^2 dx=2$

Solution : Let $f=x^2-(y')^2+\lambda y^2$

$$\frac{\partial f}{\partial y}=2\lambda y, \qquad \frac{\partial f}{\partial y'}=-2y'$$

Euler's equation becomes $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

$$2\lambda y-\frac{d}{dx}(-2y')=0$$

$$y''+\lambda y=0$$

The A.E. is $m^2+\lambda=0 \Rightarrow m=\pm\sqrt{\lambda}i$

C.F. is $e^{0x}[a\cos\sqrt{\lambda}x+b\sin\sqrt{\lambda}x]$

$$\therefore y=a\cos\sqrt{\lambda}x+b\sin\sqrt{\lambda}x$$

When $x=0, y=0 \Rightarrow 0=a$

$$x=1,\ y=0 \Rightarrow 0=a\cos\sqrt{\lambda}+b\sin\sqrt{\lambda}$$

$$0=b\sin\sqrt{\lambda} \quad \because a=0$$

$$0=\sin\sqrt{\lambda} \qquad \ldots (1)$$

Given $\int_0^1 y^2 dx=2$

$$b^2\int_0^1 \sin^2\sqrt{\lambda}\,dx=2$$

$$\frac{b^2}{2}\int(1-\cos 2\sqrt{\lambda}x)dx$$

$$\frac{b^2}{2}\left[x-\frac{\sin 2\sqrt{\lambda}x}{2\sqrt{\lambda}}\right]_0^1=1$$

$$\frac{b^2}{2}\left[1-\frac{\sin 2\sqrt{\lambda}}{2\sqrt{\lambda}}\right]=1 \qquad \ldots (2)$$

Solving (1) and (2) $b=\pm 2\sqrt{\lambda}=n\pi$

$$\therefore y=\pm 2\sin(n\pi x) \qquad \ldots (3)$$

(3) is the Extremal function for the given conditions.

4. Find the plane curve of fixed perimeter which encloses maximum area.

Solution : Let l be the perimeter of the closed surface between the points $x = x_1$ and $x = x_2$

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\,dx \qquad \text{. . . (1)}$$

The area is given by

$$A = I = \int_{x_1}^{x_2} y dx \qquad \text{. . . (2)}$$

we have to maximize (2) using (1)

consider f = y, g = $g = \sqrt{1+(y')^2}$

$$H = f + \lambda g = y + y + \lambda\sqrt{1+(y')^2}$$

Euler's Equation becomes

$$\frac{\partial H}{\partial y} - \frac{d}{dy}\left(\frac{\partial H}{\partial y'}\right) = 0$$

$$1 - \frac{d}{dx}\left[\frac{\lambda y'}{\sqrt{1+(y')^2}}\right] = 0$$

$$1 - \lambda\left\{\frac{\sqrt{1+(y')^2}.y'' - y'\dfrac{(y')y''}{\sqrt{1+(y')^2}}}{1+(y')^2}\right\}$$

$$1 - \lambda\left\{\frac{(1+(y')^2).y'' - (y')^2 y''}{[1+(y')^2]^{3/2}}\right\}$$

$$1 - \frac{\lambda y''}{(1+(y')^2)^{3/2}} = 0$$

$$\frac{1}{\lambda} = \frac{y''}{(1+(y')^2)^{3/2}}$$

$$\frac{\left[1+(y')^2\right]^{3/2}}{y''} = \lambda$$

⇒ The Radius of curvature is a constant

⇒ We know that the surface is a circle. Thus the curve with given perimeter which encloses maximum area is a circle.

5. Find the curve of fixed length πa joining (-a, 0) and (a, 0) and lying above the x – axis such that the area enclosed by it and the x – axis is maximum.

Solution : As above $A = \int_{-a}^{a} y dx =$ Area

$$\text{Length } I = \int_{-a}^{a} ds = \int_{-a}^{a} \sqrt{1+(y')^2}\,dx$$

and $H = y + \lambda\sqrt{1+(y')^2}$.

Circle will have maximum area.

Exercise

1. Find the plane curve of length λ having end points (x_1, y_1) and (x_2, y_2) such that the area is maximum.
2. Find the closed plane curve of given perimeter which encloses maximum area.
3. Find the extremal of the functional $\int_0^4 y dx$ under the constraint $\int_0^4 (y')^2 dx = 4$ given y(0) = 0 , y(4) = 4.
4. Find the value of the extremal $\int_0^2 (y')^2 dx$ under the constraint $\int_0^2 y dx = 1$ given (0, 0) (2, 1)

5. Find the extremal of the functional $\int_{-2}^{2} ydx$ given $\int_{-2}^{2} \sqrt{+(y')^2}\, dx = 2\pi$ given that y(2) = 0, y(-2) = 0

6. Find the extremal of the functional $\int_{0}^{-1} [(y')^2 + x^2]dx$ given $\int_{0}^{1} ydx = 2, \quad y(0) = 0, \quad y(1) = 1.$

7. Find the equation of a plane curve on which a particle in the absence of friction will slide from one point to another in the shortest time under the action of gravity.

8. Find the extremal of the functional $\int_{0}^{2} (y')^2 dx$ under the constraint $\int_{0}^{2} 3ydx = 2$, given y(0) = 0, y(2) = 1.

Answers

1) $(x-a)^2 + (y-b)^2 = \lambda^2$ 2) curvature = c, circle having max area. 3) $x^2 = 4y$

4) $4y = 8x - 3x^2$ 5) $x^2 + y^2 = 4$

6) $y = \cos ech2.\cosh 2x$ 10) y = $\dfrac{\sinh(\sqrt{\lambda}x)}{\sinh\sqrt{\lambda}}$

11) $y = \dfrac{1}{4}[2(1-\cos x) + (2-\pi)\sin x]$ 12) $y = 1 + 2x - 3x^2$