VIJAYA COLLEGE
RV ROAD, BASAVANAGUDI
BANGALORE - 04

BRIDGE COURSE FOR I SEMESTER STUDENTS OF B.Sc

DEPARTMENT OF MATHEMATICS

RECAPITULATION OF BASIC CONCEPTS

We recapitulate several basic concepts, formulae and results which are necessary for the future study in Mathematics.

- ALGEBRA

- **SOME BASIC ALGEBRAIC FORMULAE:**

1.$(a + b)^2 = a^2 + 2ab + b^2$.

2.$(a - b)^2 = a^2 - 2ab + b^2$.

3.$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

4.$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

5.$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

6.$(a + b + c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3b^2a + 3c^2a + 3c^2b + 6abc$.

7.$a^2 - b^2 = (a + b)(a - b)$.

8.$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

9.$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

10.$(a + b)^2 - (a - b)^2 = 4ab$.

11.$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$.

12.If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

- **Theory of indices**

The word ‘indices’ is the plural form of the word ‘index’. In a^m , m is called the index and a is called the base. a^m is called a power of a and it obeys the following rules known as the laws of indices.

(i) $a^m . a^n = a^{m+n}$, (ii) $\frac{a^m}{a^n} = a^{m-n}$, (iii) $(a^m)^n = a^{mn}$, (iv) $a^{-m} = \frac{1}{a^m}$,
(v) $a^0 = 1$ (vi) $\sqrt[n]{a}$ (i.e., nth root of a) = $a^{\frac{1}{n}}$ (vii) $a^m = a^n$ then m = n.

- **Logarithms**

If $a^x = y$, then we say that the logarithm of y to the base a is x and is written as $\log_a y = x$. If the base is 10,the logarithm is called as the common logarithm and if the base is ‘e’ (known as the exponential constant whose value is approximately2.70 the logarithm is called as the natural logarithm.

Properties of logarithms

(i) $\log_a mn = \log_a m + \log_a n$, (ii) $\log_a \frac{m}{n} = \log_a m - \log_a n$, (iii) $\log_a a = 1$, (iv) $\log_a 1 = 0$ (v) $\log_a m^n = n \log_a m$, (vi) $\log_b a = \frac{1}{\log_a b}$; $\log_a x = \frac{\log_k x}{\log_k a}$, k being a new base.

NOTE: $\log x$ means $\log_e x$. (natural logarithm)

- **Theory of equations**

The quadratic equation $ax^2 + bx + c = 0$, the values of x satisfying the equation are called as the roots or zeroes of the equation.

We have the formula for the roots of the above quadratic equation given by

$$x = \frac{-b \pm \sqrt{b^{2-4ac}}}{2a}$$

Properties:

(i) Sum of the roots $= {}^{-b}/_{a}$ and product of the roots $= {}^{c}/_{a}$

(ii) if $b^2 - 4ac$ is >0, the roots are real and distinct.

=0, the roots are real and coincident.

<0, the roots are imaginary and we denote $i = \sqrt{-1}$ or $i^2 = -1$.

Square roots of unity: -1 and 1 are square roots of unity.

Cube roots of unity: 1, $\omega = \dfrac{-1+i\sqrt{3}}{2}$, $\omega^2 = \dfrac{-1-i\sqrt{3}}{2}$ are cube roots of unity.

where $\boxed{1+\omega+\omega^2 = 0 \text{ and } \omega^3 = 1}$

Fourth roots of unity: -1, 1, i, -i are fourth roots of unity

Method of synthetic division

This method is useful in finding the roots of some equations which are of degree greater than 2. The method is illustrated through examples.

Example 1: Solve: $x^3 + 6x^2 + 11x + 6 = 0$

>> We have to first find a root by inspection which is done by taking value 1, -1, 2, -2 etc. for x and identify the value which satisfies the equation. In this example, putting x = - 1

we get – 1 + 6 -11 + 6 = 0. ∴ x = - 1is a root by inspection. We then proceed as follows to find the remaining roots.

-1	1	6	11	6	→All the coefficient are written.
	0	-1	-5	-6	→zero to be written bellow first coefficient & added
	1	5	6	0	→Resultant to be multiplied with the root -1 and added

The existing figures 1, 5, 6 are to be associated with the quadratic: $1.x^2 + 5.x + 6 = 0$ or $(x + 2)(x + 3) = 0 \rightarrow x = -2, -3$.Thus $x = -1, -2 - 3$. are the roots of the given equation.

Example 2 : Solve: $x^3 - 5x^2 + 4 = 0$
>> x = 1 is a root by inspection. Next we have as follows.

1	1	-5	0	4
	0	1	-4	-4
	1	-4	-4	0

$\therefore\ x^2 - 4x - 4 = 0$

By the quadratic formula

$x = \frac{-(-4)\pm\sqrt{16+16}}{2} = \frac{4\pm\sqrt{32}}{2} = \frac{4\pm4\sqrt{2}}{2} = 2(1 \pm \sqrt{2})$

Thus 1, $2(1 \pm \sqrt{2})$ are the roots of the given equation.

- **Progressions**

(i) The sequence $a, a + d, a + 2d$……is called as the Arithmetic Progression (A.P) whose general terms or the nth term is given by $u_n = a + (n - 1)d$ and sum to n terms is given by $s_n = \frac{n}{2}[2a + (n - 1)d]$ where ‘a’ is the first term and ‘d’ is the common difference.

(ii) The sequence a, ar, ar^2 ... . is called as the geometric Progression (G.P) whose general term is given by ar^{n-1} and sum to n term is given by $s_n = \frac{a(1-r^n)}{1-r}$ or $s_n = \frac{a(r^n-1)}{r-1}$ according as r < 1 or r > 1 where ‘a’ is the first term and ‘r’ is the common ratio. Also sum to infinity of this geometric series when r <1 is a/(1-r).

(iii) The sequence $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$ ... ... is called as the Harmonic Progression (H.P) whose general term is given by $\frac{1}{a+(n-1)d}$. In other words a sequence is said to be an H.P if its reciprocals are in A.P.
(iv) If a ,A ,b are in A.P; a, G, B are in G.P and a, H, b are in H.P then A, G, H are respectively called as the Arithmetic Mean (A.M), Geometric Mean (G.M), Harmonic Mean (H.M) beteew a and b. Their value are as follows.

$A = \frac{a+b}{2}$, $G = \sqrt{ab}$, H = $\frac{2ab}{a+b}$.

- **Mathematical Induction**

  This is a process of establishing certain results valid for positive integral values of n. In this process we first verify the result when n=1 and then assume the result to be true for some positive integer k. Later we prove the result for n=k+1. The following results established by the principle of mathematics induction will be useful.

  (i) 1+2+3+4+.........+n = Σn = $\frac{n(n+1)}{2}$.

  (ii) $1^2 + 2^2 + 3^2 + \cdots \ldots . + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$.

  (iii) $1^3 + 2^3 + 3^3 + \cdots \ldots . + n^3 = \sum n^3 = \frac{n^2(n+1)^2}{4}$.

- **Combinations**

  The notation ${}^nC_r$ means, the number of combinations of n things taken r at a time. That is selecting r things out of n things which is equivalent to rejecting (n-r) things.

  $\therefore$ ${}^nC_r = {}^nC_{n-r}$ where $n \geq r$

  The notation $n!$ read as factorial n means the product of first n natural numbers.

  i.e., $n! = 1.2.3.4 \ldots \ldots . (n-1)(n)$. It may be noted that $0! = 1$

  We also have the formula: ${}^nC_r = \frac{n!}{r!(n-r)!}$. ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$

  In particular, it is convenient to remember that ${}^nC_0 = 1$, ${}^nC_1 = n,$ ${}^nC_2 = \frac{n(n-1)}{2!}$,

  ${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$ …. Etc

  We also have the binomial theorem for a positive integer n given by

$$(x+a)^n = x^n + {}^nC_r x^{n-1} a + {}^nC_r x^{n-2} a^2 + \cdots \ldots + a^n$$

$$(x - a)^n = x^n - nC_1\, x^{n-1}\, a + nC_2\, x^{n-2}\, a^2 - nC_3\, x^{n-3}\, a^3 + \text{------------} + (-1)^r\, nC_n\, a^n.$$

- **Matrices**

A matrix is a rectangular arrangement of elements enclosed in a square or a round bracket. If a matrix has m rows and columns then the order of the matrix is said to be m × n and in particular if m = n the matrix is called a square matrix of order n.

The transpose of a matrix A is the matrix obtained by interchanging their rows and columns and is usually denoted by A'. Also a square matrix A is said to be symmetric if $A = A'$ and skew symmetric if $A = -A'$.

A square matrix having only non zero elements in its principal diagonal and zero elsewhere is called a diagonal matrix. In particular if the non zero elements are equal to 1, the matrix is called an identity matrix or unit matrix denoted by I .A matrix having all its elements equal to zero is called a null matrix or zero matrix.

Example 1: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ are diagonal matrices.

Example 2: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices.

Two matrices of the same order can be added or subtracted by adding or subtracting the corresponding elements. If $A$is any matrix and k is a constant then the matrix kA is the matrix obtained by multiplying every element of A by k.

The product of two matrices A and B can be found, if A is of order m × n and B is of order n × p.

The product AB will be a matrix of order m × p obtained by multiplying and adding the row element of A with the corresponding column elements of B.

Example 1: Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, $B = \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \\ e_1 & e_2 \end{bmatrix}$

A is of order 2×3, B is of order 3×2 ∴ AB is of order 2×2

Now $AB = \begin{bmatrix} (a_1c_1 + a_2d_1 + a_3e_1) & (a_1c_2 + a_2d_2 + a_3e_2) \\ (b_1c_1 + b_2d_1 + b_3e_1) & (b_1c_2 + b_2d_2 + b_3e_2) \end{bmatrix}$

If A and B are two square matrices such that $\mathrm{AB} = \mathrm{I}$, then B is called as the inverse of A denoted by A^{-1}. Thus $\mathrm{AA}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}$.

- **Determinants**

The determinant is denoted for a square matrix A. It is usually denoted by $|A|$ and will represent a single value on expansion as follows.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

i.e, $= a_1\,(b_2c_3 - b_3c_2) - a_2\,(b_1c_3 - b_3c_1) + a_3\,(b_1c_2 - b_2c_1)$

Important properties of determinants

(i) The determinant can be expanded through any row or column as above keeping in mind the sings $+, -, +, -$ etc. as we move from one element to the other starting from the first row or column.

(ii) The value of the determinant remains unaltered if the rows and columns are interchanged.

(iii) The value of the determinant is zero if any two rows or columns are identical or proportional.

(iv) The property of common factors is presented below.

$$\begin{vmatrix} ka_1 & ka_2 \\ b_1 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \; ; \begin{vmatrix} ka_1 & a_2 \\ kb_1 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Determinant method to solve a system of equations (Cramer's rule)

Consider,
$$a_1x + a_2y + a_3z = k_1$$
$$b_1x + b_2y + b_3z = k_2$$
$$c_1x + c_2y + c_3z = k_3$$

Let $\Delta_x = \begin{vmatrix} k_1 & a_2 & a_3 \\ k_2 & b_2 & b_3 \\ k_3 & c_2 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & k_1 & a_3 \\ b_1 & k_2 & b_3 \\ c_1 & k_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & a_2 & k_1 \\ b_1 & b_2 & k_2 \\ c_1 & c_2 & k_3 \end{vmatrix}$ & $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Then $x = \frac{\Delta_x}{\Delta}$, $y = \frac{\Delta_y}{\Delta}$, $z = \frac{\Delta_z}{\Delta}$. Here Δ is the coefficient determinant and $\Delta_x, \Delta_y, \Delta_z$ are the determinants obtained by replacing the coefficient of x ,y ,z respectively with that of constants in the R.H.S of the given equations.

Rule of cross multiplication

If ,

$$a_1x + a_2y + a_3z = 0$$
$$b_1x + b_2y + b_3z = 0$$

Then the proportionate values of x, y, z ie., x: y: z are given by $\frac{x}{\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$.

- **Partial fractions**

This is a method employed to convert an algebraic fraction $f(x)/g(x)$ into a sum. The basic

requirement is that the degree of the numerator must be less than the degree of the denominator , in which case the fraction is said to be a proper fraction. If $f(x)/g(x)$ is proper fraction we factorize $g(x)$ and resolve into partial fractions appropriately which is based on the nature of the factors present in the denominator.

We have the following cases.

(i) $\frac{f(x)}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)} + \frac{C}{(x-\gamma)}$.

(ii) $\frac{f(x)}{(x-\alpha)^2(x-\beta)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\alpha)^2} + \frac{C}{(x-\beta)}$.

(iii) $\frac{f(x)}{(x-\alpha)(px^2+qx+r)} = \frac{A}{(x-\alpha)} + \frac{Bx+c}{(px^2+qx+r)}$.

Where $(px^2 + qx + r)$is a non factorizable quadratic and A ,B, C are all consants to be found by first simplifying R.H.S taking L.C.M. Later we take convenient values for x to obtain A, B ,C constants A, B, C can also be evaluated by comparing the coefficient of various powers of x in L.H.S and R.H.S.

If $f(x)/g(x)$ is an improper fraction $[degree\ of\ f(x) \geq degree\ of\ g(x)]$ we divide and rewrite using the known concept that,

$$\frac{f(x)}{g(x)} = (Quotient) + \frac{Remainder}{Divisor} = Q + \frac{F(x)}{G(x)}$$

Here $\frac{F(x)}{G(x)}$ will be a proper fraction .

NOTE: It is important to note that if needed a non factorizable quadratic $(px^2 + qx + r)$ can be factorized involving constants like $a + ib$ or $a + \sqrt{b}$ (in the factors0. Which being the roots of $(px^2 + qx + r)$= 0.

Observe the following.

(i) $(x^2 + 25) = (x + 5i)(x - 5i)$

(ii) $(x^2 - 5) = (x + \sqrt{5)}\ (x - \sqrt{5})$

(iii)$(x^2 - 2x - 1) = (x - 1 + \sqrt{2})$ Here $(1 \pm \sqrt{2})$ are the roots of $(x^2 - 2x - 1) = 0$.

Following results are useful

i) $\frac{f(x)}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{f(a)}{x-a} - \frac{f(b)}{x-b}\right)$

ii) $\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$ iii) $\frac{1}{(x^2-a^2)(x^2-b^2)} = \frac{1}{a^2-b^2}\left(\frac{1}{x^2-a^2} - \frac{1}{x^2-b^2}\right)$

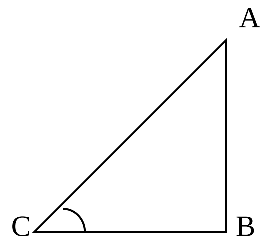
❖ <u>TRIGONOMETRY</u>

Area of a sector of a circle = $\frac{1}{2}r^2\theta$.

Arc length, S = $r\,\theta$.

- **Trigonometric ratios**

  Let ABC be a right angled triangle and let $A\hat{C}B = \theta$. The side AB is called as the opposite side of θ, BC is the adjacent side and AC is the hypotenuse. The six trigonometric ratios are defined as follows.



$$sin\theta = \frac{AB}{AC} \quad cos\theta = \frac{BC}{AC} \quad tan\theta = \frac{AB}{BC} \quad cosec\theta = \frac{AC}{AB} \quad sec\theta = \frac{AC}{BC} \quad cot\theta = \frac{AC}{AB}$$

(The expanded form of these are respectively sine, cosine, tangent, cosecant, secant and cotangent,)

Inter relations:

$$sin\theta = \frac{1}{cosec\theta} \;;\; cos\theta = \frac{1}{sec\theta} \;;\; tan\theta = \frac{1}{cot\theta} \;;\; tan\theta = \frac{sin\theta}{cos\theta}.$$

$$cosec\theta = \frac{1}{sin\theta} \;;\; sec\theta = \frac{1}{cos\theta} \;;\; cot\theta = \frac{1}{tan\theta} \;;\; cot\theta = \frac{cos\theta}{sec\theta}.$$

Identities:

(i) $sin^2\theta + cos^2\theta = 1$

(ii)1+$tan^2\theta = sec^2\theta$

(iii) 1+$cot^2\theta = cosec^2\theta$

- **Radian measure:** $\pi\ radians = 180^o$ (Degrees can be converted into radians and vice-versa).

Example 1: $45^o = 45 \times \frac{\pi}{180} = \frac{\pi}{4} radians$

Example 2: $\frac{2\pi}{3} = \frac{2\times180}{3} = 120^o$

Trigonometric ratios for certain standard angles

θ	0^o	30^o $(\pi/6)$	45^o $(\pi/4)$	60^o $(\pi/3)$	90^o $(\pi/2)$	180^o (π)	360^o (2π)	15^0 $(\pi/12)$	75^0 $(5\pi/12)$
$sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
$cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
$tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	0	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$

$cot\theta, sec\theta, cosec\theta$ are respectively the reciprocals of $tan\theta, cos\theta, sin\theta$.

- **Allied angles**

Trigonometrical ratios of $90 \pm \theta, 180 \pm \theta, 270 \pm \theta, 360 \pm \theta$ in terms of those of θcan be found easily by the following rule known as A –S –T –C rule.

(i) When the angle is $90 \pm \theta \ or \ 270 \pm \theta$ the trigonometrical ratio changes from sine to cosine and vice-versa. Also ‘tan’ & ‘cot’, ‘sec’ & ‘cosec’.
(ii) When the angle is $180 \pm \theta, 360 \pm \theta$ the trigonometrical ratio remains the same… i.e, sin→sin, cos→cos etc.
(iii) In each case the sign + or – is premultiplied by the A –S –T –C quadrant rule.

S II $(90^o - 180^o)$	A I $(0^o - 90^o)$
T III $(180^o - 270^o)$	C IV $(270^o - 360^o)$

A : All ratios are +ve in the I quadrant.

S : ‘Sin’ is + ve in the II quadrant.

T : ‘Tan’ is + ve in the III quadrant.

C : ‘Cos’ is + ve in IV quadrant.

NOTE: $\sin(-\theta) = -sin\theta$, $\cos(-\theta) = cos\theta$, $\sin(n2\pi + \theta) = sin\theta$, $\cos(n2\pi + \theta) = cos\theta$

Example 1: $\sin(90^o - \theta) = cos\theta, \cos(90^o + \theta) = -sin\theta$

$\sin(180 - \theta) = sin\theta, \tan(180 + \theta) = tan\theta$.

Example 2: $\sin(135^o) = \sin(90^o + 45^o) = cos45^o = 1/\sqrt{2}$

$\tan(315^o) = \tan(270^o + 45^o) = -cot45^o = -1$

$\cos(225^o) = \cos(180^o + 45^o) = -cos45^o = -1/\sqrt{2}$

$\sin(750^o) = \sin(2 \times 360^o + 30^o) = sin30^o = 1/2$

- **Compound angle formulae**

(i) $\sin(A + B) = sinA\ cosB + cosA\ sinB$

$\text{Sin}(A - B) = sinA\ cosB - cosA\ sinB$

(ii) $\cos(A + B) = cosA\ cosB - sinA\ sinB$

$\cos(A - B) = cosA \cos B + sinA\ sinB$

(iii) $\tan(A + B) = \frac{tanA + tanB}{1 - tanAtanB}$

$\text{Tan}(A - B) = \frac{tanA - tanB}{1 + tanAtanB}$

Formulae to convert a product into sum or difference

(iv) $sinA\ cosB = \frac{1}{2}\ [\sin(A + B) + \sin(A - B)]$

(v) $cosA\ sinB = \frac{1}{2}\ [\sin(A + B) - \sin(A - B)]$

(vi) $cosA\ cosB = \frac{1}{2}\ [\cos(A + B) + \cos(A - B)]$

(vii) $sinA\ sinB = -\frac{1}{2}\ [\cos(A + B) - \cos(A - B)]$

Particular cases of formulae

- $sin2A = 2sinAcosA$
- $sinA \quad = 2\sin(A/2)\cos(A/2)$
- $cos2A = cos^2A - sin^2A$
- $cos2A = 1 - 2sin^2A$
- $cos2A = 2cos^2A - 1$
- $cosA = cos^2(A/2) - sin^2(A/2\ )$
- $cosA = 1 - 2sin^2\ A/2$
- $cosA = 2cos^2\ A/2 - 1$
- *tan 2A* $= \frac{2\tan A}{1-\tan^2 A}$,
- $tanA = \frac{2tanA/2}{1-tan^2A/2}$
- $sin3A = 3sinA - 4sin^3A$
- $cos3A = 4cos^3A - 3cosA$
- *tan 3A*= $\frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$.

We have, $\quad sin2A = \frac{2tanA}{1+tan^2A} \quad or \ \ sinA = \frac{2tanA/2}{1+tan^2A/2}$

$$cos2A = \frac{1-tan^2A}{1+tan^2A} \quad or\ cosA = \frac{1-tan^2A/2}{1+tan^2A/2}$$

Almost used formulae in problem:

$1 + \cos 2A = 2\cos^2 A \ or \cos^2 A = \frac{1}{2}(1+\cos 2A)$.

$1 - \cos 2A = 2\sin^2 A \ or \ \sin^2 A = \frac{1}{2}(1-\cos 2A)$,

$1 + \sin 2A = (\sin A + \cos A)^2$ *and* $1 - \sin 2A = (\cos A - \sin A)^2 = (\sin A - \cos A)^2$,

Formulae to convert a sum or difference into a product

(i) $sinC + sinD = 2sin\frac{C+D}{2}\ cos\frac{C-D}{2}$

(ii) $sinC - sinD = 2cos\frac{C+D}{2}\ sin\frac{C-D}{2}$

(iii) $cosC + cosD = 2cos\frac{C+D}{2}\ cos\frac{C-D}{2}$

(iv) $cosC - cosD = -2sin\frac{C+D}{2}\ sin\frac{C-D}{2}$

- **Relation between the sides and angles of a triangle**

  In any triangle ABC a, b ,c respectively denotes the sides AC, CA, AB. These three sides along with angles $\hat{A}, \hat{B}, \hat{C}$ from the six elements of the triangle. We give two important formulae relating these elements.

  (i) **Sine formula**: If ABC is a triangle inscribed in a circle then $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2r, r$ being the circum radius.

  (ii) **Cosine formula** : In any triangle ABC, $a^2 = b^2 + c^2 - 2bc\ cosA$; $b^2 = c^2 + a^2 - 2ac\ cosB$; $c^2 = a^2 + b^2 - 2ab\ cosC$.

  (iii)**Projection Rule:** a = b cosC +c cosB

  b = c cosA +a cosC

  c = a cosB +b cosA

  (iv)**Tangents Rule:** $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$,

  $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right)$,

  $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$.

Half angle formulae:

$\sin\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\left(\frac{A}{2}\right)=\sqrt{\frac{s(s-a)}{bc}}$, $\tan\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

$\sin\left(\frac{B}{2}\right)=\sqrt{\frac{(s-a)(s-c)}{ac}}$, $\cos\left(\frac{B}{2}\right)=\sqrt{\frac{s(s-b)}{ac}}$, $\tan\left(\frac{B}{2}\right)=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$.

$\sin\left(\frac{C}{2}\right)=\sqrt{\frac{(s-a)(s-b)}{ab}}$, $\cos\left(\frac{C}{2}\right)=\sqrt{\frac{s(s-c)}{ab}}$, $\tan\left(\frac{C}{2}\right)=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$.

Area of triangle ABC = $\sqrt{s(s-a)(s-b)(s-c)}$, where 2s = a + b + c.

Area of triangle ABC = $\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B=\frac{1}{2}ab\sin C$.

In any triangle ABC, we have

1. sin2A + sin2B + sin2C = 4sinAsinBsinC
2. sin2A + sin2B - sin2C = 4cosAcosBsinC
3. cos2A + cos2B + cos2C = -1- 4cosAcosBcosC
4. cos2A + cos2B - cos2C = 1- 4sinAsinBcosC

Inverse Trigonometric functions:

Function	Domain	Range

y = sin^{-1}x	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \mathbf{y} \le \frac{\pi}{2}$
y = cos^{-1}x	$-1 \le x \le 1$	$0 \le \mathbf{y} \le \pi$
y = tan^{-1}x	$-\infty \le x \le \infty$	$-\frac{\pi}{2} \le \mathbf{y} \le \frac{\pi}{2}$
y = cot^{-1}x	$-\infty \le \mathbf{x} \le \infty$	$0 < \mathbf{y} < \pi$
y = sec^{-1}x	$x \le -1 \ or \ x \ge 1$	$0 \le y \le \pi$, **except** $y \ne \frac{\pi}{2}$
y = cosec^{-1}x	$x \le -1 \ or \ x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, **except** $y \ne 0$

General Solution of Trignometric Equations:

1. Sin θ = k**, where -1≤ k ≤ 1**
   Sin θ = sin α, where k = **sin** α

Gen. solun. is $\boxed{\theta = n\pi + (-1)^n \alpha}$

2. cos θ = k**, where -1≤ k ≤ 1**
   cos θ = cos α, where k = **sin** α

Gen. solun. is $\boxed{\theta = 2n\pi \pm \alpha}$

3. tan θ = k**, where -∞ ≤ k ≤ ∞**

tan θ = tan α , where k = **tan** α

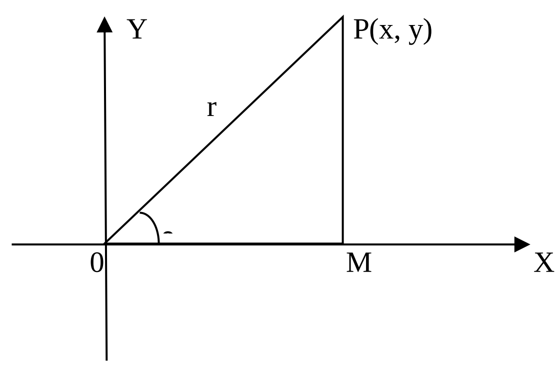
Gen. solun. is $\boxed{\theta = n\pi + \alpha}$ **where n € Z**

All the cases n = 0, 1, 2, 3, 4,---

❖ COMPLEX TRIGONOMETRY

➢ A number of the from $z = x + iy$ where x, y are real numbers and $i = \sqrt{-1}$ or $i^2 = -1$ is called a complex number in the Cartesian form x is called the real part of z and y is called imaginary part of z. $\bar{z} = x - iy$ is called the complex conjugate of z.

Polar form of $z = x + iy$: The point (x , y) is plotted in the XOY plane and let OP= r, PM is drawn perpendicular onto the x – axis.



From the figure OM= x, PM = Y, $P\hat{O}M = \theta$

Further $cos\theta = \frac{x}{r}$, $sin\theta = \frac{y}{r}$ i.e., $x = r\ cos\theta, y = r\ sin\theta$.

Squaring & adding : $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$

Dividing : $tan\theta = y/x\ or\ \theta = \tan^{-1}(y/x)$.

$z = x + iy = r(cos\theta + i\ sin\theta)$ is called as the polar form of z. further it may be noted that $e^{i\theta} = cos\theta + i\ sin\theta$. thus $z = re^{i\theta}$ is the polar form of z where r is called the ‘modulus’ of z and θ is called the ‘amplitude’ or ‘argument of z. Symbolically we have,

$r = |z| = \sqrt{x^2 + y^2}$; $\theta = amp\ z = \arg z = \tan^{-1} y/x$.

Also we have $e^{-i\theta} = cos\theta - i\ sin\theta$ and hence we can write, $e^{i\theta} + e^{-i\theta} = 2cos\theta$ and
$e^{i\theta} - e^{-i\theta} = 2i\ sin\theta$.

- **De-Moivre's Theorem :** If n is a rational number (positive or negative integer ,fraction) then $(cos\theta + i\ sin\theta)^n = \cos n\theta + i \sin n\theta$.
- **Expansion of $sin^m x$, $cos^m x, sin^m x cos^n x$ where m, n are positive integers**

The method is explained through two examples as it becomes easy to grasp the procedure involved.
Example 1: To expand $cos^6 x$
>> Let $t = cosx + i\ sinx$ $\therefore\ t^n = \cos nx + i \sin nx$

Also $\frac{1}{t} = cosx - i\ sinx$ $\therefore \frac{1}{t^n} = \cos nx - i \sin nx$
Adding and subtracting these we get,
$t + \frac{1}{t} = 2cosx \ldots\ldots\ldots (1)$ $t - \frac{1}{t} = 2i \sin x \ldots\ldots. (2)$
$t^n + \frac{1}{t^n} = 2cosnx \ldots\ldots (3)$
$t^n - \dfrac{1}{t^n} = 2i \sin nx \ldots\ldots (4)$
NOTE : Forming equation (1) to(4) is a common step in all the expansions mentioned.

To expand $cos^6 x$, we need to rise (1) to the power 6 and write it in the form

$$(2\ cosx)^6 = \left(t + \frac{1}{t}\right)^6$$

Expanding R.H.S by the binomial theorem we have,

$$2^6 cos^6 x = t^6 + {}^6C_1 t^5.\frac{1}{t} + {}^6C_2 t^4.\frac{1}{t^2} + {}^6C_3 t^3.\frac{1}{t^3} + {}^6C_4 t^2.\frac{1}{t^4} + {}^6C_5 t.\frac{1}{t^5} + \frac{1}{t^6}$$

But $\quad {}^6c_5 = {}^6c_1 = 6\,; \quad {}^6c_4 = {}^6c_2 = \frac{6\times5}{1\times2} = 15\,; \quad {}^6c_3 = \frac{6\times5\times4}{1\times2\times3} = 20$

$$\therefore 2^6 cos^6 x = \left(t^6 + \frac{1}{t^6}\right) + 6\left(t^4 + \frac{1}{t^4}\right) + 15\left(t^2 + \frac{1}{t^2}\right) + 20$$

Using equation (3) in the R.H.S by taking n = 6, 4, 2 we have,

$$2^6 cos^6 x = 2\cos 6x + 6(2\,cos4x) + 15\,(2cos2x) + 20$$

$$\therefore \boldsymbol{cos^6 x = \frac{1}{2^5}\,(\cos 6x + 6\cos 4x + 15\cos 2x + 10)}$$

Example 2: To expand $sin^5 x\, cos^2 x$

>> Rise (2) to the power 5, (1) to the power 2 and multiply.

$\therefore (2isinx)^5(2cosx)^2 = \left(t - \frac{1}{t}\right)^5\left(t + \frac{1}{t}\right)^2$

i.e., $(2^5 i^5 sin^5 x)(2^2 cos^2 x) = \Big(t^5 - {}^5C_1 t^4.\frac{1}{t} + {}^5C_2 t^3.\frac{1}{t^2} - {}^5C_3 t^2.\frac{1}{t^3} + {}^5C_4 t.\frac{1}{t^4} - \frac{1}{t^5}\Big) \times \left(t^2 + \frac{1}{t^2} + 2\right)$

Here $\quad i^5 = i^2 \times i^2 \times i = -1 \times -1 \times i = i$

But $\quad {}^5c_4 = {}^5c_1 = 5\,,\ {}^5c_3 = {}^5c_2 = \frac{5\times4}{1\times2} = 10.$

Thus we have, $2^7 isin^5 xcos^2 x = \left(t^7 - \frac{1}{t^7}\right) - 3\left(t^5 - \frac{1}{t^5}\right) + \left(t^3 - \frac{1}{t^3}\right) + 5\left(t - \frac{1}{t}\right)$

Using (4) in the R.H.S by taking n=7, 5, 3 and 1 we have

$$2^7 isin^5 xcos^2 x = 2i\sin 7x - 3(2i\,sin5x) + 2isin3x + 5\,(2i\,sinx)$$

Dividing by 2i throughout we get,

$$\therefore \boldsymbol{sin^5 xcos^2 x = \frac{1}{2^5}\,(\sin 7x - 3\sin 5x + \sin 3x + 5\,sinx)}$$

❖ HYPERBOLIC FUNCTIONS

We have already said that 'e' whose value is approximately 2.7 is called the exponential constant. Further if $\log_e y = x$ then $y = e^x$ is called the exponential function. Hyperbolic function are defined in terms of exponential functions as follows.

Sine hyperbolic of x = $\sinh x = \frac{e^x - e^{-x}}{2}$.

Cosine hyperbolic of x = $\cosh x = \frac{e^x + e^{-x}}{2}$.

Also, $\tanh x = \frac{\sinh x}{\cosh x}$; $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$

$$\operatorname{sech} x = \frac{1}{\cosh x} \ ; cosech\ x = \frac{1}{\sinh x}$$

Important hyperbolic identities

(i) $cosh^2\theta - sinh^2\theta = 1$

(ii)1+$tan^2\theta = sec^2\theta$

(iii) 1+$cot^2\theta = cosec^2\theta$

(iv) $sinh^2\theta + cosh^2\theta = \cosh 2\theta$

(v)$2 \sinh\theta \cosh\theta = \sinh 2\theta$

Relationship between trigonometric and hyperbolic functions

We have $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ & $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

Now $\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = (-1)\frac{(e^x - e^{-x})}{2i} = i^2 \frac{e^x - e^{-x}}{2i}$

i.e., $\sin(ix) = i . \frac{e^x - e^{-x}}{2}$ $\quad \therefore sin(ix) = i\, sinh\, x$

Also $\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh x$ $\quad \therefore \cos ix = coshx$

From these we can deduce the remaining four relations in respect of tan(ix), cot(ix) sec(ix) & cosec(ix).

- Calculus

- **Function**

Let x and y be two variables which take only real values. If there is a relation between x and y such that for a given x it is possible to find a corresponding value of y, then y is said to be a function of x which is symbolically represented as y = f(x). This means that y depends on x. x is called the independent variable and y is called the dependent variable.

For example y = f(x) = 3x +1 means that y is a function of x.
If x = 0 then y = f(0) = 3×0+1 = 1;
If x =2, y = f(2) = 3×2 +1 = 7 etc.
For every x we get only one corresponding value of y. Such a function is called a single valued function.

Now consider the Example: $y^2 = x^2 + 4$ or $y = \pm\sqrt{x^2+4}$
When x = 0, y = ± $\sqrt{4}$ ie., y = +2 or y = -2
When x = 1, y = ± $\sqrt{5}$ ie., y = +$\sqrt{5}$ or y = -$\sqrt{5}$ etc.
For every x we get more than one of y.
Such a function is called a many valued function.

Further if y = 3x+1 then y -1= 3x or x = $\frac{1}{3}$ (y-1).
Also in the case of $y^2 = x^2 + 4$ we have $x^2 = y^2 - 4$ or $x = \pm\sqrt{y^2 - 4}$.
It should be observed that in these two examples we have expressed x in terms of y and such a function is called an inverse function symbolically represent as x = f^{-1}(y) read as, x is equal to f inverse y.
$f[f^{-1}(x)] = x = f^{-1}[f(x)]$. Observe the following functions and their corresponding inverse function.
(i) $siny = x$ or $y = \sin^{-1} x$
(ii) $tany = x$ or $y = \tan^{-1} x$
(iii) $coshy = x$ or $y = cosh^{-1}x$
(iv)$\log_e y = x$ or $y = e^x$
It may be noted that $\sin(\sin^{-1} x), \tan^{-1}(tanx) = x, \log(e^x) = x = e^{logx}$ etc.

- **Limits**

Consider y = f(x) and when x =a, y = f(a). sometimes f(a) assumes forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty, 0^0$, 1^∞ etc which are all undefined, In mathematics these are called indeterminate forms.

Observe the example $y = \frac{x^2-1}{x-1}$

When x = 1 y = $\frac{0}{0}$ which is undefined.

We can rewrite y in the form, y = $\frac{(x-1)(x+1)}{(x-1)}$

Suppose x ≠ 1we can cancel the factors (x-1).

∴ When x = 1, y = $\frac{0}{0}$ and when x ≠ 1, y = x+1.

Instead of taking x = 1 we shall give values for x which are very near to 1 (little less than 1 or little more than 1) and tabulate them:

x	0.9	0.99	0.999	1.1	1.01	1.001	…..
y = x+1	1.9	1.99	1.999	2.1	2.01	2.001	……

From the table it can be observed that when the value of x is little less or more than 1, the value of y is little less or more than 2. But y has no value when x = 1. This is equivalent to saying that then when x is closer and closer to 1, y is closer and closer to 2. In other words we say that the limit of y as x tends to 1 is equal to 2 and is symbolically represented as follows

$\lim_{x \to 1} y = 2$ or $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

Also if y = $\frac{1}{x}$ it is obvious that as x increases y decreases and this can be represented in the limit form as $\lim_{x \to \infty} \frac{1}{x} = 0$

The following are some of the established standard limits.

(i) $\lim_{x \to a} \left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}$, n is any rational number

(ii) $\lim_{x\to 0}\frac{sinx}{x} = 1$

(iii) $\lim_{x\to 0}\frac{tanx}{x} = 1$

(iv) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

(v) $\lim_{x\to\infty} x^{1/x} = 1$

(vi) $\lim_{x\to 0}(1+x)^{1/x} = e$ or $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Simple illustrations

Example 1: $\lim_{x\to 2}\frac{x^5-32}{x-2}$ this is in the $\frac{0}{0}$ form.

$= \lim_{x\to 2}\frac{x^5-32}{x-2} = 5\times 2^{5-1} = 80$, using (i)

Example 2: $\lim_{x\to 0}\frac{sinax}{sinbx}\ldots\ldots\frac{0}{0}$ form

$= \lim_{x\to 0}\frac{\frac{sinax}{x}}{\frac{sinbx}{x}} = \lim_{x\to 0}\frac{a\left(\frac{sinax}{ax}\right)}{b\left(\frac{sinbx}{bx}\right)} = \frac{a\times 1}{b\times 1} = \frac{a}{b}$ by using (ii)

Example 3: $\lim_{n\to\infty}\frac{3n+4}{5n+1}\ldots.\frac{\infty}{\infty}$ form

$= \lim_{n\to\infty}\frac{n\left(3+\frac{4}{n}\right)}{n\left(5+\frac{1}{n}\right)} = \frac{3+0}{5+0} = \frac{3}{5}$ $\quad\because \frac{1}{n}\to 0$ as $n\to\infty$

- **Differentiation**

Let y = f(x) be a continuous explicit function of x. Any change in x result in a corresponding change in y. Let x change to x + δx and let the corresponding change in y be y + δy. δx, δy are respectively called the increments in x, y.

∴ we have y = f(x) and y + δy = f(x + δx)

Also y +δy –y = f(x +δx) –f(x)

i.e., δy = f(x+δx) –f(x)

Then $\lim_{\delta x\to}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ when exists is called as the differential coefficient or the derivative of y with respect to x (w.r.t.x) and is denoted by $\frac{dy}{dx}$ or $f'(x)$ here $\frac{d}{dx}$ is called the differential operator (+, -, ×, ÷) usually denoted by D. $\frac{dy}{dx}$ means $\frac{d}{dx}(y)$ which is to be understood as the derivative of y w.r.t x.

Example: Let $y = x^n$, n is any real number

$\therefore y + \delta y = (x + \delta x)^n$

Subtracting we get $\delta y = (x + \delta x)^n - x^n$

$$\lim_{\delta x \to} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x + \delta x)^n - x^n}{\delta x}$$

As δx → 0, x + δx → x. putting t = x+ δx we have,

$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{t^n - x^n}{t - x} = nx^{n-1}$,By using standard limit (i)

Thus $f'(x) = \frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$.

The following table gives a list of derivatives established for standard functions.

Differentiation Formulae	**Differentiation of Functions**
1. $\frac{d(x^n)}{dx} = nx^{n-1}$	$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} . \frac{d}{dx}(f(x))$
2. $\frac{d(e^x)}{dx} = e^x$	$\frac{d}{dx}[e^{f(x)}] = e^{f(x)} . \frac{d}{dx}(f(x))$
3. $\frac{d}{dx}(a^x) = \log a . a^x$	$\frac{d}{dx}[a^{f(x)}] = a^{f(x)} . \frac{d}{dx}(f(x))$
4. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$	$\frac{d}{dx}[\log_e f(x)] = \frac{1}{f(x)} . \frac{d}{dx}(f(x))$
5. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}(\sqrt{f(x)}) = \frac{1}{2\sqrt{f(x)}} . \frac{d}{dx}(f(x))$

6. $\frac{d}{dx}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$ $\qquad \frac{d}{dx}\left(\frac{1}{f(x)}\right)=-\frac{1}{[f(x)]^2}\frac{d}{dx}(f(x))$

7. $\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}$ $\qquad \frac{d}{dx}\left(\frac{1}{[f(x)]^n}\right)=-\frac{n}{[f(x)]^{n+1}}\frac{d}{dx}(f(x))$

8. $\frac{d}{dx}(\sin x)=\cos x$ $\qquad \frac{d}{dx}(\sin f(x))=\cos(f(x)).\frac{d}{dx}(f(x))$

9. $\frac{d}{dx}(\cos x)=-\sin x$ $\qquad \frac{d}{dx}(\cos f(x))=-\sin(f(x)).\frac{d}{dx}(f(x))$

10. $\frac{d}{dx}(\tan x)=\sec^2 x$ $\qquad \frac{d}{dx}(\tan f(x))=\sec^2(f(x)).\frac{d}{dx}(f(x))$

11. $\frac{d}{dx}(\cot x)=-co\sec^2 x$ $\qquad \frac{d}{dx}(\cot f(x))=-co\sec^2(f(x)).\frac{d}{dx}(f(x))$

12. $\frac{d}{dx}(\sec x)=\sec x\tan x$ $\qquad \frac{d}{dx}(\sec f(x))=\sec(f(x))\tan(f(x)).\frac{d}{dx}(f(x))$

13. $\frac{d}{dx}(co\sec x)=-co\sec x\cot x$ $\qquad \frac{d}{dx}(co\sec f(x))=-co\sec f(x)\cot(x)\frac{d}{dx}(f(x))$

14. $\frac{d}{dx}(\sinh x)=\cosh x$ $\qquad \frac{d}{dx}(\sinh f(x))=\cosh(f(x)).\frac{d}{dx}(f(x))$

15. $\frac{d}{dx}(\cosh x)=\sinh x$ $\qquad \frac{d}{dx}(\cosh f(x))=\sinh(f(x)).\frac{d}{dx}(f(x))$

16. $\frac{d}{dx}(\tanh x)=\sec h^2 x$ $\qquad \frac{d}{dx}(\tanh f(x))=\sec h^2(f(x)).\frac{d}{dx}(f(x))$

17. $\frac{d}{dx}(\coth x)=-co\sec h^2 x$ $\qquad \frac{d}{dx}(\coth f(x))=-co\sec h^2(f(x)).\frac{d}{dx}(f(x))$

18. $\frac{d}{dx}(\sec hx)=-\sec hx\tanh x$, $\qquad \frac{d}{dx}(\sec hf(x))=-\sec h(f(x))\tanh(f(x)).\frac{d}{dx}(f(x))$

19. $\frac{d}{dx}(cosechx)=-cosechx\ cothx$ $\qquad \frac{d}{dx}(cosechf(x))=-cosechf(x)cothf(x).\frac{df(x)}{dx}$

20. $\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}\left(\sin^{-1}f(x)\right)=\frac{1}{\sqrt{1-f(x)^2}}.\frac{d}{dx}\left(f(x)\right)$

21. $\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}\left(\cos^{-1}f(x)\right)=\frac{-1}{\sqrt{1-f(x)^2}}.\frac{d}{dx}\left(f(x)\right)$

22. $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}\left(\tan^{-1}f(x)\right)=\frac{1}{1+f(x)^2}.\frac{d}{dx}\left(f(x)\right)$

23. $\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}\left(\cot^{-1}f(x)\right)=\frac{-1}{1+f(x)^2}.\frac{d}{dx}\left(f(x)\right)$

24. $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$ $\qquad$ $\frac{d}{dx}\left(\sec^{-1}f(x)\right)=\frac{1}{f(x)\sqrt{f(x)^2-1}}.\frac{d\left(f(x)\right)}{dx}$

25. $\frac{d}{dx}\left(co\sec^{-1}x\right)=-\frac{1}{x\sqrt{x^2-1}}$ $\qquad$ $\frac{d}{dx}\left(co\sec^{-1}f(x)\right)=\frac{-1}{f(x)\sqrt{f(x)^2-1}}.\frac{d\left(f(x)\right)}{dx}$

26. $\frac{d}{dx}\left(\sinh^{-1}x\right)=\frac{1}{\sqrt{1+x^2}}$ $\qquad$ $\frac{d}{dx}\left(\sinh^{-1}f(x)\right)=\frac{1}{\sqrt{1+f(x)^2}}.\frac{d}{dx}\left(f(x)\right)$

27. $\frac{d}{dx}\left(\cosh^{-1}x\right)=-\frac{1}{\sqrt{x^2-1}}$ $\qquad$ $\frac{d}{dx}\left(\cosh^{-1}f(x)\right)=\frac{1}{\sqrt{f(x)^2-1}}.\frac{d}{dx}\left(f(x)\right)$

28. $\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}$ $\qquad$ $\frac{d}{dx}\left(\tanh^{-1}f(x)\right)=\frac{1}{1-f(x)^2}.\frac{d}{dx}\left(f(x)\right)$

29. $\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{1-x^2}$ $\qquad$ $\frac{d}{dx}\left(\coth^{-1}f(x)\right)=\frac{1}{1-f(x)^2}.\frac{d}{dx}\left(f(x)\right)$

30. $\frac{d}{dx}\left(\sec h^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}}$ $\qquad$ $\frac{d}{dx}\left(\sec h^{-1}f(x)\right)=\frac{-1}{f(x)\sqrt{1-f(x)^2}}.\frac{d\left(f(x)\right)}{dx}$

31. $\frac{d}{dx}\left(co\sec h^{-1}x\right)=\frac{-1}{x\sqrt{x^2+1}}$ $\qquad$ $\frac{d}{dx}\left(co\sec h^{-1}f(x)\right)=\frac{-1}{f(x)\sqrt{f(x)^2+1}}.\frac{d\left(f(x)\right)}{dx}$

- **Rule of differentiation**

Rule – 1: Function of a function rule or chain rule

If y = f(u) where u = g(x) then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = f'(u).g'(x)$$

In other words, if y = f[g(x)] then $\frac{dy}{dx} = f'[g(x)].g'(x)$

Further if y = f[g{h(x)}] then $\frac{dy}{dx} = f'[g\{h(x)\}].g'\{h(x)\}.h'(x)$

Example: $y = \log(sinx)$, $y = \sin(m\sin^{-1}x), y = \tan^{-1}e^{3x}$, $y = e^{m\cos^{-1}x}$

Rule – 2: Product rule

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ ie., $(uv)' = uv' + vu'$

Example: $y = e^x \log x$, $y = \sqrt{\sin x}\,.\tan(\log x)$.

Rule – 3: Quotient rule

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ ie., $\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

Example: $y = \frac{\log x}{x}$, $y = \frac{1+sin3x}{1-sin3x}$.

- **Differentiation of implicit functions**

If x and y are connected by a relation then the process of finding the derivative of y w.r.t x is called as the differentiation oh implicit function.

In other words, given f(x, y) = c we need to find $\frac{dy}{dx}$ treating y as a function of x.

We have already said that $\frac{d}{dx}[f\{g(x)\}]$ is equal to $f'\{g(x)\}.g'(x)$. This is equivalent to saying that

$$\boldsymbol{\frac{d}{dx}[f(y) = f'(y)\frac{dy}{dx}}$$

Example:$xsiny + x^2 + y^2 = ycosx$,$x^2 + y^2 + r^2$.

- **Differentiation of parametric functions**

If x and y are function of a parameter t we need to find the derivative of y w.r.t. x ie., given x = f(t) , y = g(t) we have to find $\frac{dy}{dx}$

The rule is $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ and it is obvious that $\frac{dy}{dx}$ is a function of t.

Example: $x = at^2$& $y = 2at, x = a(cos\theta + \theta sin\theta)$ &
$y = a(sin\theta - \theta cos\theta)$

- **Logarithmic differentiation**

Suppose we have the explicit function in the form $y = [f(x)]^{g(x)}$ or an implicit function in the form $[f(x)]^{g(y)} = c$ (say)then we have to first take logarithms on bothsides and then differentiate w.r.t x for computing $\frac{dy}{dx}$ If the index is a variable then it becomes necessary to employ logarithmic differentiation for computing the derivative. Sometimes we employ this method otherwise also to make the differentiation work simpler.

Example: $y = a^x, r^n = a^n cosn\theta$

- **Differentiation by using trigonometric substitution and formulae**

Sometimes the given function will be in such a form that it will be difficult to differentiate in the direct approach. In such cases we have to think of a suitable substitution leading to a trigonometric formulae that simplifies the given function considerably, thereby we can differentiate easily.

Example: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right), y = \tan^{-1}\left(\frac{1-x}{1+x}\right)$ put $x = \tan\theta$

- INTEGRATION

Integration is regarded as the reverse process of differentiation (anti differentiation). For example, we know that $\frac{d}{dx}(\sin x) = \cos x$. This is equivalent to saying that the integral of $\cos x$ with respect to x is equal to $\sin x$. Since the derivative of a constant is always zero we can as well write, $\frac{d}{dx}(\sin x + c) = \cos x$ or equivalently integral of $\cos x$ w.r.t. x is $\sin x + c$. The symbol $\int$ stands for the integral. Thus in general we can say that if $\frac{d}{dx}[f(x) + c] = F(x)$ then $\int F(x)dx = f(x) + c$, c being the arbitrary constant.

A list of Integration of some standard functions

$\int x^n dx = \frac{x^{n+1}}{n+1}$, where $n \neq -1$	$\int \operatorname{cosec} ax\, dx = \frac{\log(\operatorname{cosec} ax - \cot ax)}{a}$
$\int \frac{1}{x} dx = \log x$	$\int \sec^2 ax\ dx = \frac{\tan ax}{a}$
$\int e^x dx = e^x$	$\int \operatorname{cosec}^2 ax\, dx = -\frac{\cot ax}{a}$
$\int a^x dx = \frac{a^x}{\log a}$	$\int \sec ax \tan ax\, dx = \frac{\sec ax}{a}$
$\int \sin x dx = -\cos x$	$\int \operatorname{cosec} ax \cot ax\, dx = -\frac{\operatorname{cosec} ax}{a}$
$\int \cos x dx = \sin x$	$\int \sinh x\ dx = \cosh x$
$\int \tan x\ dx = \log(\sec x) = -\log \cos x$	$\int \cosh x\ dx = \cosh x$
$\int \cot x\ dx = \log(\sin x)$	$\int \tanh x\ dx = \log \cosh x$
$\int \sec x\ dx = \log(\sec x + \tan x)$	$\int \operatorname{sech}^2 x\ dx = \tanh x$
$\int \operatorname{cosec} x\ dx = \log(\operatorname{cosec} x - \cot x)$	$\int \operatorname{cosech}^2 x\, dx = -\coth x$
$\int \sec^2 x\ dx = \tan x$	$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$
$\int \operatorname{cosec}^2 x\, dx = -\cot x$	$\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x$
$\int \sec x \tan x\, dx = \sec x$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x = -\cos^{-1} x$

$\int co\sec x \cot x\, dx = -co\sec x$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x = -\cot^{-1} x$
$\int \sin ax\ dx = -\frac{\cos ax}{a}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x = -\cos ec^{-1} x$
$\int \cos ax\ dx = \frac{\sin ax}{a}$	$\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x$
$\int \tan ax\ dx = \frac{\log(\sec ax)}{a} = -\frac{\log\cos ax}{a}$	$\int \frac{1}{\sqrt{x^2-1}}\ dx = \cosh^{-1} x$
$\int \cot ax\ dx = \frac{\log(\sin ax)}{a}$	$\int \frac{1}{x\sqrt{1-x^2}} dx = -\sec h^{-1} x$
$\int \sec ax\ dx = \frac{\log(\sec ax + \tan ax)}{a}$	

Note: The above results can easily be verified by differentiating the R.H.S of the results.

- **Some methods of integration**

Method 1: $\int \frac{f'(x)}{f(x)} dx = \log f(x)$. If the numerator is the derivative of the denominator then the integral is equal to logarithm of the denominator.

Further if $\frac{d}{dx}[f(x)] = kf'(x)$, where k is a constant then

$$\int \frac{f'(x)}{f(x)} dx = \frac{1}{k}\int \frac{kf'(x)}{f(x)} dx = \frac{1}{k}\log f(x).$$

Example: $\int \frac{1}{3x+4} dx$, $\int \frac{1}{tanx(1+x^2)} dx$, $\int \frac{1}{x\log x} dx$, $\int \frac{2x+3}{3x+1}$

Method 2: Integration by substitution

In this method a suitable substation is taken to reduce the given integral to a simpler form in the new variable and integrate accordingly. Taking substation is only by judgment in case of simple problems, though we have specific substitutions for some specific form of integrals.

Example: $\int \frac{(3x+4)dx}{\sqrt{3x^2+8x+5}}$, $\int \frac{x^2}{\sqrt{1-x^6}} dx$.

Remark: Trigonometric function, hyperbolic functions will also serve as substitutions in the evaluation of certain standard integrals and these integrals are given later. They will be highly useful.

Method 3: Integration by parts (Product rule)

$$\int uv\, dx = u \int v\, dx - \int \int v\, dx\, .u'\ dx$$

It should be noted that this rule is not applicable for integrals of product of any two functions. However if the first function is a polynomial in x and the integral of the second function is known the rule can be tried. In such a case we also have a generalized product rule known as the Bernoulli's rule which is as follows.

$$\int uv\, dx = u \int v\, dx - u' \int \int v\, dx\, dx + u'' \int \int \int v\, dx\, dx\, dx - \cdots ..$$

Example: $\int x cos 3x\, dx, \int \log x\, dx, \int (x^3 + x + 1)e^{2x} dx.$

Method 4: Method of partial fractions

The method is narrated in the Algebra section. After resolving the function $f(x)/g(x)$ into partial functions we have to integrate term by term.

Method 5: Integration by using trigonometric formulae

Various integrals involving products of sine and cosine terms and terms like $sin^2 x, cos^2 x, sin^3 x, cos^3 x$ etc. can be found by using various trigonometric formulae as given in the Trigonometry section. These formulae converts product into a sum.

Example: $\int cos^2 3x\, dx, \int sin 4x\, cos 2x\, dx.$

Standard integrals

With the help of the method discussed earlier, the following standard integrals are obtained and they will be highly useful.

1) $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}[a\sin(bx+c)+b\cos(bx+c)]$

2) $\int e^{ax}\cos(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}[a\cos(bx+c)-b\sin(bx+c)]$

Note: (1) &(2) are obtained by parts.

Integration by substitution

1. $\int [f(x)]^n . f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}$

2. $\int (ax+b)^n dx = \dfrac{1}{a}\dfrac{(ax+b)^{n+1}}{n+1}$

3. $\int \sin(ax+b)\,dx = -\dfrac{1}{a}\cos(ax+b)$

4. $\int \cos(ax+b)\,dx = \dfrac{1}{a}\sin(ax+b)$

5. $\int \tan(ax+b)dx = \dfrac{1}{a}\log\left(\sec(ax+b)\right) = -\dfrac{1}{a}\log\left(\cos(ax+b)\right)$

6. $\int \cot(ax+b)\,dx = \dfrac{1}{a}\log\left(\sin(ax+b)\right)$

7. $\int \sec(ax+b)\,dx = \dfrac{1}{a}\log\left(\sec(ax+b)+\tan(ax+b)\right)$

8. $\int \operatorname{cosec}(ax+b)\,dx = \dfrac{1}{a}\log\left(\operatorname{cosec}(ax+b)-\cot(ax+b)\right)$

9. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$ (by putting x = a tanθ)

10. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)$, where $a > x$ (by partial fractions)

11. $\int \frac{\mathbf{dx}}{\mathbf{x}^2 - \mathbf{a}^2} = \frac{1}{\mathbf{2a}} \mathbf{log}\left(\frac{\mathbf{x - a}}{\mathbf{x + a}}\right)$, where $\mathbf{a} < \mathbf{x}$ (by partial fractions)

12. $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{a}^2 - \mathbf{x}^2}} = \mathbf{sin}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$ (by putting x = a sinθ)

13. $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{a}^2 + \mathbf{x}^2}} = \mathbf{sinh}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$ (by putting x= a sinhθ)

14. $\int \frac{\mathbf{dx}}{\sqrt{\mathbf{x}^2 - \mathbf{a}^2}} = \mathbf{cosh}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$ (by putting x = a coshθ)

15. $\int \sqrt{\mathbf{a}^2 - \mathbf{x}^2}\,\mathbf{dx} = \frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a}^2 - \mathbf{x}^2} + \frac{\mathbf{a}^2}{\mathbf{2}}\mathbf{sin}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$ (by putting x = a sinθ)

16. $\int \sqrt{\mathbf{a}^2 + \mathbf{x}^2}\,\mathbf{dx} = \frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a}^2 + \mathbf{x}^2} + \frac{\mathbf{a}^2}{\mathbf{2}}\mathbf{tan}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$ (by putting x = a sinhθ)

17. $\int \sqrt{\mathbf{x}^2 - \mathbf{a}^2}\,\mathbf{dx} = \frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{x}^2 - \mathbf{a}^2} - \frac{\mathbf{a}^2}{\mathbf{2}}\mathbf{cosh}^{\mathbf{-1}}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)$(by putting x = a coshθ)

Note: $\sin^{-1} x/a = \log(x + \sqrt{x^2 + a^2})$

$\cos^{-1} x/a = \log(x + \sqrt{x^2 - a^2})$

We now proceed to give a few method where the standard integrals are being used.

Method 6: Integration by completing the square

This method is applicable for integrals of the type:

$$1/_{ax^2 + bx + c},\ 1/_{\sqrt{ax^2 + bx + c}},\ \sqrt{ax^2 + bx + c.}$$

We consider $ax^2 + bx + c$ and write it in the form $a(x^2 + bx/a + c/a)$. We then express it in the form $a[(x \pm \alpha)^2 \pm \beta^2]$, α and β being constants, we complete the integration by making use the standard integrals.

Example: $\int \frac{dx}{x^2+6x+25}, \int \frac{dx}{\sqrt{x^2+6x+25}}, \int \sqrt{x^2+6x+25}dx.$

Method 7: Integral of the types:

$$\frac{px+q}{ax^2+bx+c}, \frac{px+q}{\sqrt{ax^2+bx+c}}, (px+q)\sqrt{ax^2+bx+c}$$

In all the three cases we first express the numerator (linear term) as *l* (derivative of the quadratic) + *m*. where *l*, *m* are constants to be found. That is to find l and m such that $px+q = l(2ax+b)+m$

Example: $\int \frac{(4x+5)dx}{4x^2+12x+5}, \int \frac{(4x+5)dx}{\sqrt{4x^2+12x+5}}, \int (4x+5)\sqrt{4x^2+12x+5}dx.$

- A few more types of integral along with substitution for the purpose of integration is as follows.

1. $\int \frac{dx}{(ax+b)\sqrt{cx+d}} \ldots . put\ (cx+d) = t^2$

2. $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}} \ldots . put = \frac{1}{t}$

3. $\int \frac{dx}{a\ sinx+b\ cox+c} \ldots . put\ \tan(x/2) = t$ and use the result $sinx = \frac{2t}{1+t^2}$, $cosx = \frac{1-t^2}{1+t^2}$ Also *dx* will become $\frac{2dt}{1+t^2}$

4. $\int \frac{dx}{a^2cox^2x+b^2sin^2x+c}$ .... multiply and divide by sec^2x and then put *tanx* = *t*

In all the above cases we arrive at a quadratic in t.

- **Definite integrals**

  We define $\int_{x=a}^{x=b} f(x)dx = g(b) - g(a)$ where $g(x) = \int f(x)dx$

Observe the following examples

$\int_1^2 x^2 dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

$\int_0^{\pi/2} cosx\, dx = [sinx]_0^{\pi/2} = \sin(\pi/2) - sin0 = 1 - 0 = 1.$

Properties of Definite Integral

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$

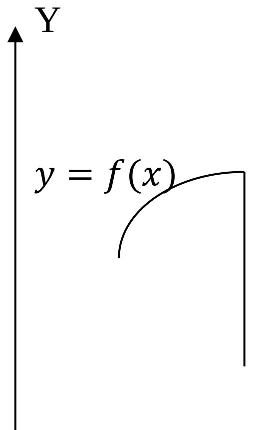
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

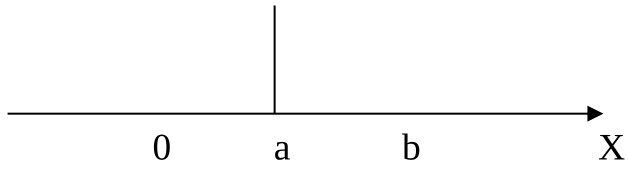
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where a < c < b

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \quad f(x)\, is\, even \quad i,.e \quad f(-x) = f(x) \\ 0 & if \quad f(x)\, is\, odd \quad i,.e \quad f(-x) = -f(x) \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & if \quad f(2a-x) = f(x) \\ 0 & if \quad f(2a-x) = -f(x) \end{cases}$

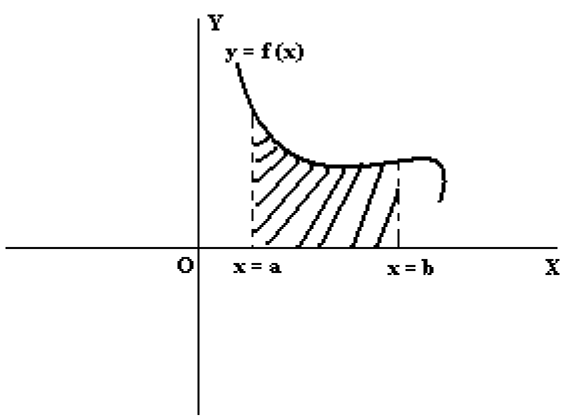




Note: Geometrically $\int_a^b f(x)dx$ represents the area bounded by the curve $y = f(x)$, the x – axis and the ordinates x = a, x = b.

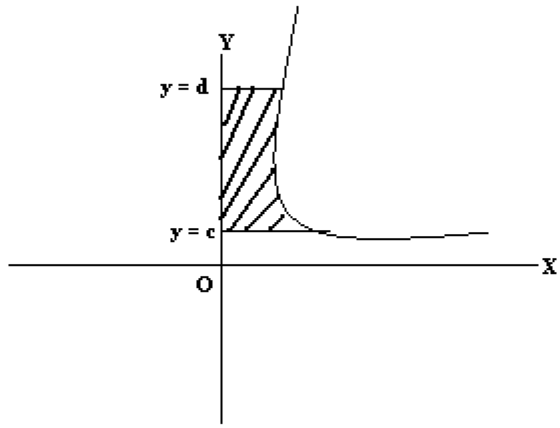
- **Application of Integration**

I. Area bounded by the curve y = f(x), X-axis and x = a, x = b is given by



$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx$$

II. Area bounded by the curve x = g(y), Y-axis and y = c, y = d is given by

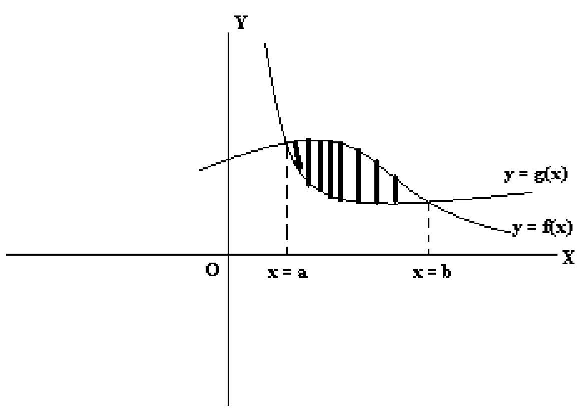


Area = $\int_a^b x\,dy = \int_a^b g(y)dy$

III. Area bounded by the curve y = f(x), y = g(x), X-axis and x = a, x = b is given by

Or

Area bounded between the curves y = f(x), y = g(x) intersecting at x = a, x = b is given by



Area = $\int_a^b f(x)\,dx - \int_a^b g(x)dx$

❖ VECTOR ALGEBRA

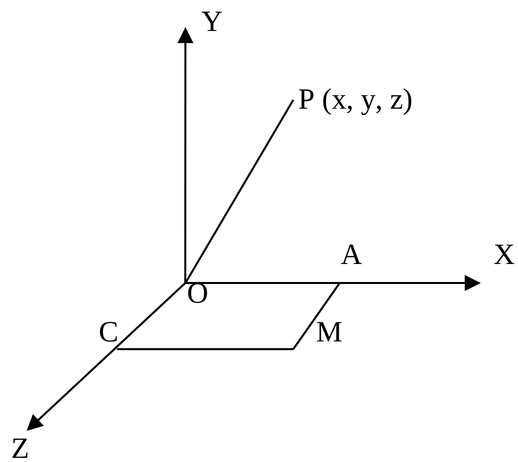
Vector is a quantity having both magnitude and direction. Scalar is a quantity having only magnitude. For example, force, velocity, acceleration are vector quantities. Density, mass are scalar quantities.

A vector with magnitude equal to one is called a unit vector. The line segment joining a given point to the origin is called as the position vector of that point.

- **Representation of a vector in three dimensions**

Let OX and OY be two mutually perpendicular straight lines. Draw a line OZ perpendicular to the XOY plane. In other words OX, OY, OZ are said to from three mutually perpendicular straight lines. Let P be any point in space and from P draw PM perpendicular to the XOY plane. Also draw MA perpendicular to the X – axis, MC perpendicular to the Z – axis and PB perpendicular to the Y – axis.

If OA = x, OB = y, OC = z then the coordinates of P = (x, y, z).



Suppose we take three points respectively on the coordinate axes at a distance of 1 unit from the origin O then these will have coordinates respectively (1, 0, 0), (0, 1, 0) and (0, 0, 1) . the associated line segments are the unit vectors: $\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$ and $\hat{k} = (0,0,1)$ along the coordinate axes and are called the basic unit vectors.

We have (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)

The vector representation of the point P is written in the standard form $\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Magnitude of $\vec{r}$ = distance between O (0, 0, 0) and P (x, y, z) = $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$

$\therefore |\vec{r}| = \sqrt{x^2+y^2+z^2}$ is the magnitude of $\vec{r}$. Also $\hat{n} = \vec{r}/_{|\vec{r}|}$ is always a unit vector.

$\vec{O} = 0i + 0j + 0k$ is called the null vector.

- **Definition and properties**

(1) Dot and cross products : Let $\vec{A}$ and $\vec{B}$ be any two vectors subtending an angle θ between them. Also let $\hat{n}$ represent the unit vector perpendicular to the plane containing the vectors $\vec{A}$ and $\vec{B}$ (i.e., perpendicular to both $\vec{A}$ and $\vec{B}$). Then we have

$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|cos\theta$ (Scalar quantity)

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|sin\theta\hat{n}$ (Vector quantity) where $\vec{A}, \vec{B}, \vec{n}$ forms a right handed system.

(2) Angle between the vectors $\vec{A}$ & $\vec{B}$: From the definition of the dot product we have

$cos\theta = \frac{\vec{A}\,\vec{B}}{|\vec{A}||\vec{B}|}$ .Further the vectors are perpendicular if $\theta = \pi/_2$or

$\cos\theta = cos\,\pi/_2 = 0 => \vec{A}.\vec{B} = 0$ i.e., to say that $\vec{A}$ is perpendicular to B if $\vec{A}.\vec{B} = 0$

(3)Properties : (i) $\vec{A}.\vec{B} = \vec{B}.\vec{A}$; $\vec{A} \times \vec{B} = -(\vec{A} \times \vec{B})$

(ii) $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1$

(iii) $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

(iv) $\hat{\imath} \times \hat{\jmath} = \overrightarrow{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$

(v)$\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0$

(4) Analytic expressions for the dot and cross products

If $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

Then $\vec{A}.\vec{B} = a_1b_1 + a_2b_2 + a_3b_3$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(5) Scalar triple product or Box product
$\vec{A}.(\vec{B}\times\vec{C})$ Denoted by $[\vec{A},\vec{B},\vec{C}]$ is called as the scalar triple product or the box product of the vectors $\vec{A},\vec{B},\vec{C}$
Properties:
(i) $\vec{A}.(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B}).\vec{C}$
(ii) $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{B}\times\vec{A})$
(iii) $[\vec{A},\vec{B},\vec{C}]$ is equal to the value of coefficient determinant of the vectors $\vec{A},\vec{B},\vec{C}$
(iv) If any two vectors are identical in a box product then the value is equal to zero. In such a case we say that the vectors are coplanar.

(6) Expressions for the vector triple product
(i) $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$
(ii) $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$
Simple illustration: Given $\vec{A} = 2i + j - 2k, \vec{B} = 3i - 4k,$
Let us compute $\vec{A}.\vec{B}, \vec{A}\times\vec{B}$ and the angle between $\vec{A}$ and $\vec{B}$
>> $\vec{A}.\vec{B} = (2i + j - 2k).(3i - 4k)$
$\vec{A}.\vec{B} = (2)(3) + (1)(0) + (-2)(-4) = 14$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 3 & 0 & -4 \end{vmatrix} = i(-4-0) - j(-8+6) + k(0-3)$$
$$= -4i + 2j - 3k$$

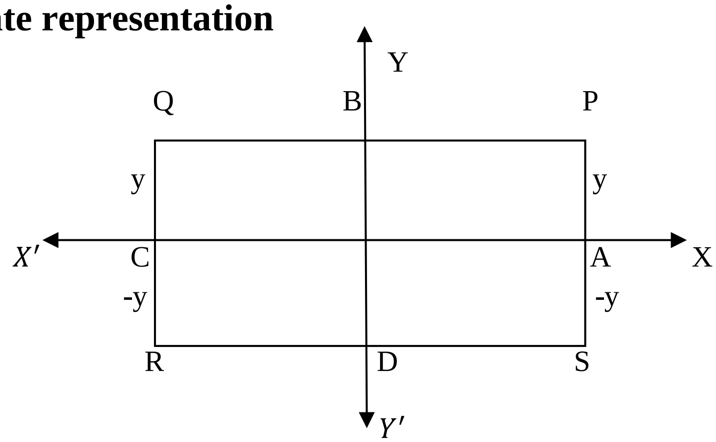
If θ is the angle between $\vec{A}$ and $\vec{B}$ then

$$cos\theta = \frac{\vec{A}\,\vec{B}}{|\vec{A}||\vec{B}|} = \frac{14}{\sqrt{2^2+1^2+(-2)^2}\sqrt{3^2+(-4)^2}} = \frac{14}{(3)(5)} = \frac{14}{15}$$

$$cos\theta = \frac{14}{15} \ or \ \cos^{-1}\left(\frac{14}{15}\right) = \theta.$$

❖ COORDINATE GEOMETRY OF TWO DIMENSIONS

➢ Coordinate representation



The coordinate representation of the points shown in the figure with reference to the origin

O = (0, 0) is as follows.
P = (x, y) Q = (- x, y) R = (- x, - y) S = (x, - y) A = (x, 0) B = (0, y)
C = (- x, 0) D = (0, - y)
Some basic formulae : Let A = (x_1, y_1) and B = (x_2, y_2)
(i) Distance AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Distance formula)
(ii) Coordinates of the point P dividing the line joining AB in the ratio $l{:}m$ is given by$\left(\frac{lx_2 \pm mx_1}{l \pm m} \ \frac{ly_2 \pm my_1}{l \pm m}\right)$ [section formula]
The sign is positive if P divides AB internally and the sing is negative if P divides AB externally.

Particular case : (a) The coordinates of the midpoint of AB is given by $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$
[$l = m$ in the external division formula]

(b) The coordinates of the point dividing AB in the ratio 1 : k internally is given by $\left(\frac{x_2+kx_1}{1+k}, \frac{y_2+ky_1}{1+k}\right)$. This coordinate is regarded as the coordinate of any arbitrary point lying on the line AB.

Locus : The path traced by a point which moved according to alaw (certain geometrical conditions) is called as the **locus.**

- **Straight line**

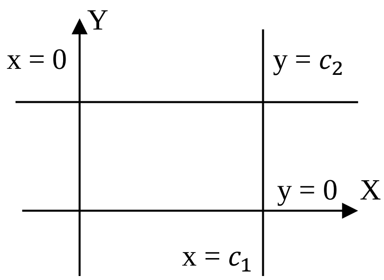
If a straight line make an angle θ with the positive direction of the x – axis then m = tanθ is called as the slope or the gradient of the straight line.

NOTE: If y = f(x) be the equation of a curve then $f'(x) = \frac{dy}{dx}$ represents the slope of the tangent at a point P(x, y) on it.

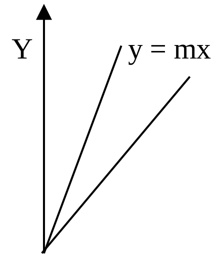
The equation representing different kinds of straight lines along with figures is as follows.

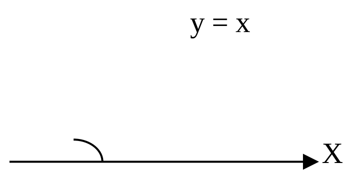
(1) x = 0 and y = 0 respectively represent the equation of the y – axis and the x – axis.

$x = c_1$ & $y = c_2$ are respectively the equations of a line parallel to the y – axis and the equation of a line parallel to the x – axis.

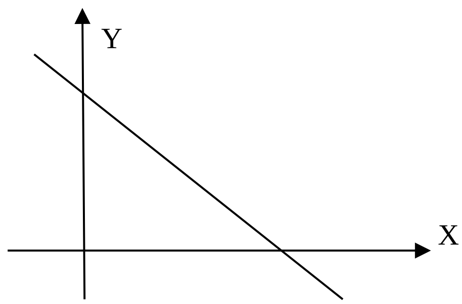


(2) y = mx is the equation of a line passing through the origin having slope m. In particular y = x is the equation of a line through the origin subtending an angle 45^o with the x – axis (slope = 1).





(3) $\frac{x}{a}+\frac{y}{b}=1$ (intercept form) is the equation of a line having x intercept a and y intercept b. That is the line passing through (a, 0) and (0, b).



(4) y = mx + c is the equation of a straight line having slope m and an intercept c on the y – axis.

(5) $y-y_1=m(x-x_1)$ is the equation of a straight line passing through the (x_1,y_1) having slope m.

(6) $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$ is the equation of a straight line passing through the points (x_1,y_1) & (x_2,y_2). Also $m=\frac{y_2-y_1}{x_2-x_1}$ is slope of a straight line joining the points (x_1,y_1) & (x_2,y_2).

(7) Ax +By +C = 0 is the equation of a straight line in the general form i.e., $y=-\frac{A}{B}x-\frac{C}{B}$ and comparing with y = mx + c we can say that the slope (m) $=-\frac{A}{B}=\frac{-\,coeff.of\ x}{coeff.of\ y}$.

Angle : Angle θ between the straight lines having slope $m_1$and m_2 is given by $tan\theta=\frac{m_1-m_2}{1+m_1m_2}$ Further the line are perpendicular if $\theta={}^{\pi}/_{2}$which implies $tan\theta=\infty$ and we must have $1+m_1m_2$ = 0 or $m_1m_2=-1$. Also if the lines are parallel then θ = 0which implies tan θ = 0 and hence we have, $m_1 - m_2 = 0 \;\; or \;\; m_1 = m_2$

'Lines are perpendicular if the product of the slope is – 1 and lines are parallel if the slopes are equal'

Length of the perpendicular: The length of the perpendicular from an external point (x_1, y_1) onto the line ax + by + c = 0 is given by $\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$.

- **Circle**

  The equation of the circle with centre (a, b) and radius r is $(x-a)^2 + (y-b)^2 = r^2$. In particular if origin is the centre of the circle then the equation is $x^2 + y^2 = r^2$

  The equation of the circle in its general form is given by, $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre is$(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$

- **Conics**

If a point moves such that its distance from a fixed point (known as the focus) bears a constant ratio with the distance from a fixed line (known as the directrix) the path traced by the point (locus of the point) is known as a conic. The constant ratio is known as the eccentricity of the conic usually denoted by 'e'. the conics are respectively called as parabola, ellipse and hyperbola according as e = 1, e < 1 and e >1. Also it may be noted that an asymptote is a tangent to the curve at infinity.

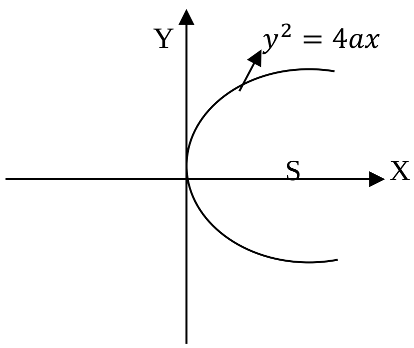
Some useful information about the conics with their equation in the Cartesian and parametric forms along with their shapes are given.

(1) Parabola

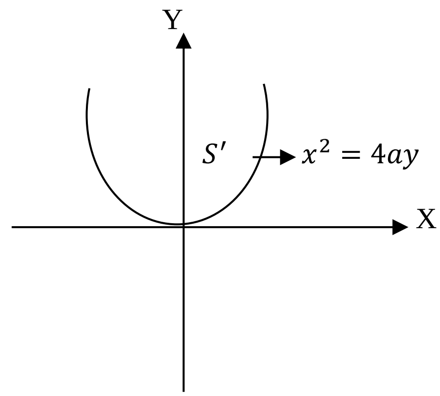
$y^2 = 4ax$ (Symmetrical about the x – axis) ; $x = at^2, y = 2at$

$x^2 = 4ay$ (Symmetrical about the y – axis) ; $y = at^2, x = 2at$

General forms of parabola : $(y-k)^2 = 4a(x-h); (x-h)^2 = 4a(y-k)$



Focus=S=(a, 0)



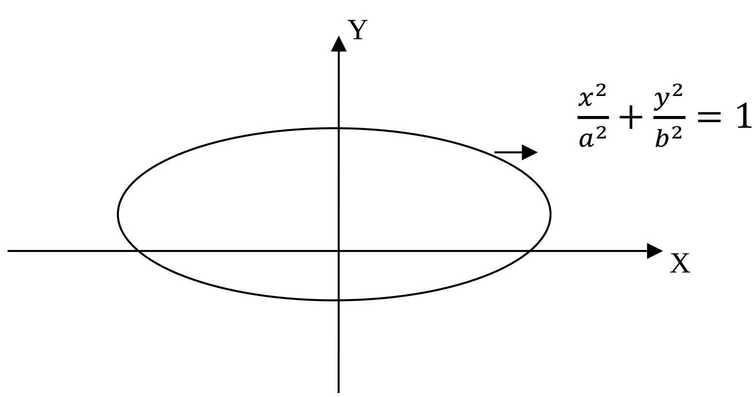
Focus=S'=(0, a)

(2) Ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $x = acos\theta, y = b\ sin\theta$

Length of the major and minor axis are respectively 2a and 2b S(Focus)=(ae, 0), S'(Focus) =(- ae, 0)

Coordinates of foci (± ae, 0) = ($\pm\sqrt{a^2 - b^2}$,0)



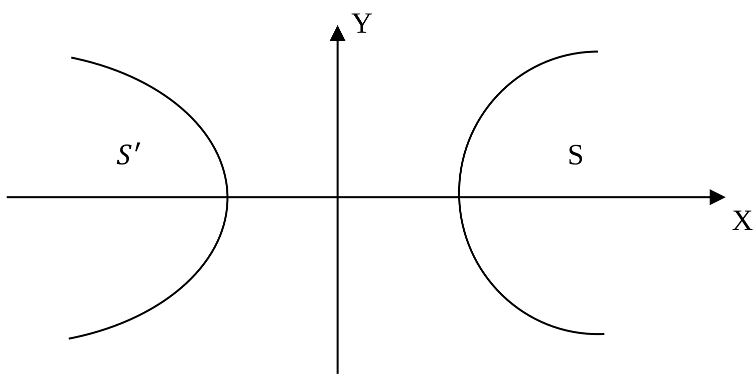
(3) Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; x = a\ sec\theta, y = b\ tan\theta$

S (Focus)=(ae, 0), S'(Focus) =(- ae, 0)

Coordinates of foci (± ae, 0) = ($\pm\sqrt{a^2 + b^2}$,0)

It may also be noted that $= c^2$;$x = ct, y = c/t$ is the equation of a curve known as the rectangular hyperbola.



❖ AREA, VOLUME, SURFACE AREA

Circle: Area = πr^2; Circumference = $2\pi r$.

Ellipse: Area = πab

Square: Area = x^2; Perimeter = 4x.

Rectangle: Area = xy ; Perimeter = 2(x + y).

Triangle: Area = $\frac{1}{2}$(base)(height) ; Perimeter = 2s = a + b + c.

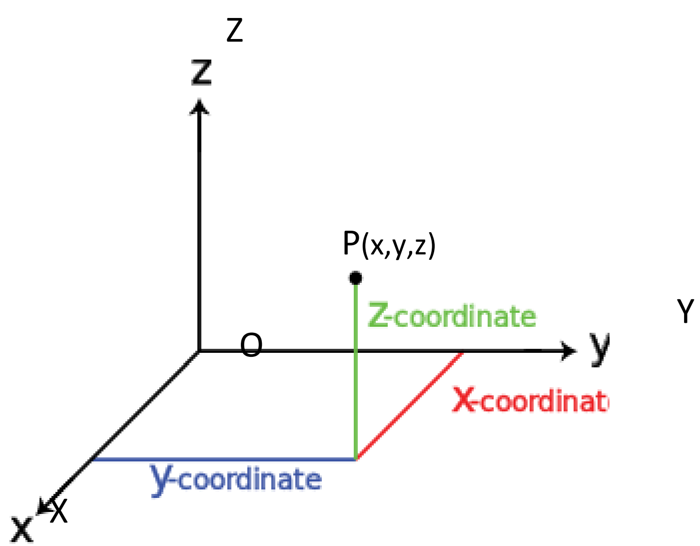
Area of equilateral triangle = $\frac{\sqrt{3}}{4}a^2$.

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Name of solid		Volume	Lateral or curved surface area	Total surface area
1.	Cube	a^3	$4a^2$	$6a^2$
2.	Cuboid	lbh	$2(l+b)h$	$2(lb+bh+hl)$
3.	Cylinder	$\pi r^2 h$	$2\pi rh$	$2\pi r(r+h)$
4.	Cone	$1/3\ .\pi r^2 h$	πrl	$\pi r(r+l)$
5.	Sphere	$4/3\ .\pi r^3$	-	$4\pi r^2$

Where in the case of cone l is the slant height connected by the relation $t^2 = r^2 + h^2$.

Three Dimensional Geometry:



Type equation here.

Direction cosines and direction ratios of a line:

If a line makes an angle α, β and γ with the co-ordinate axes, then l=cosα, m=cosβ and n=cosγ are called direction cosines of the line and a, b, c are direction ratios given by the relations

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

Direction cosines and direction ratios of a line passing through two points (x_1,y_1,z_1) & (x_2,y_2,z_2)

Directions ratios x_2-x_1, $y_2, z_2 - z_1$ and direction cosines are

$$\frac{x_2-x_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{y_2-y_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}, \frac{z_2-z_1}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

EQUATION LINE IN THREE DIMENSIONS:

I. Equation of a line thriugh $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ (vector form)

Cartesian form of the equation is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

II. Equation of a line through two points $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b}-\vec{a})$ (vector form)

Cartesian form of the equation is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

III. Angle between two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Note: 1. condition for two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are parallel is

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

2. condition for two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are parallel is

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

IV. Shortest distance between two skew lines $\vec{a}_1 + \lambda\vec{b}_1$ and $\vec{a}_2 + \lambda\vec{b}_2$ *is*

$$d=\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}$$

V. Distance between the parallel lines $\vec{a}_1 + \lambda\vec{b}$ and $\vec{a}_2 + \lambda\vec{b}$ *is*

$$d=\frac{|\vec{b}\times(\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$$

EQUATION OF PLANE:

I. Equation of a plane in normal form $\vec{r}\cdot\hat{n} = d$ (vector form)

Cartesian form of the equation is $lx + my + nz = d$, where l, m, n are d.cs of the normal and 'd' is length of the normal.

II. Equation of a plane perpendicular to the given vector and passing through a point is $(\vec{r}-\vec{a})\cdot\vec{N} = 0$ (vector form)

Cartesian form of the equation is $A(x-x_1)+B(y-y_1)+C(z-z_1)my+nz=0$, where A, B, C are d.rs of the normal $\vec{N}$

III. Equation of a plane passing through three non-collinear points $\vec{a}, \vec{b}, \vec{c}$ is $(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = 0$ (vector form)

Cartesian form of the equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$