RIGID BODIES - MOMENT OF INERTIA

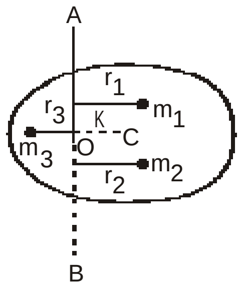
The inability of a body to change by itself its position of rest or uniform motion is called Inertia. The greater the mass of the body, the greater its inertia as greater force is required to bring about a desired change in the body. Thus the mass of the body is taken as a measure of its inertia for translatory motion.

Similarly a body, capable of rotation about an axis, possesses inertia for rotational motion. The greater the couple or torque required to change the state of rotation of the body, the greater its rotational inertia. This rotational inertia of the body is called the M.I of the body about the axis of rotation.

Thus M.I. is the rotational amalogue of mass in translatory motion.

Definition of M.I of a particle of mass 'm' about a given axis of rotation is defined as the product of the mass and the square of the distance 'r' of the particle from the axis of rotation. i.e., M.I $\boxed{I = Mr^2}$

Consider a body of mass M capable of rotation about a fixed axis AOB. The body can be imagined to be made up of a no. of particles of mass m_1, m_2, m_3,...... at distances r_1, r_2, r_3,......... from the axis of rotation. Then the M.I of the body about the axis of rotation = sum of the M.Is of all the particles about the axis of rotation



i.e., $I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... = \Sigma mr^2$

$$\Rightarrow \boxed{I = \Sigma mr^2} \text{ -----(1)}$$

Thus M.I of a body about an axis of rotation is defined as the sum of the plroducts of the mass and square of the distance of all the particles constituting the body from the axis of rotation.

RADIES OF GYRATION

Consider a body of mass M capable of rotation about an axis. Let the entire mass M of the body be imagined to be concentrated at a point C. This point is called the Centre of mass of the body. Let C be at a distance K from the axis of rotation. Then M.I. of the body about the axis of rotation can be written as $\boxed{I = MK^2}$---(2) where K is called the radius of gyration of the body w.r.t. the axis of rotation.

Def : Radius of gyration of a body capable of rotation about an axis is defined as the distance of the point where the entire mass of the body is imagined to be concentrated from the axis of rotation.

From (1) & (2) : $MK^2 = \Sigma mr^2$

$$\text{or } \boxed{K^2 = \frac{mr^2}{M}}$$

KINETIC ENERGY OF ROTATION

Consider a rigid body rotating with a constant angular speed ω about a fixed axis AOB as shwon in the figure. As the body is made up of a number of particles of masses m_1, m_2, m_3, ... at distances r_1, r_2, r_3, ... from the axis of rotation, all these particles describe circular paths of radii r_1, r_2, r_3, ... with the same angular speed ω. As the linear velocity of the particles $V = r\omega$, V is different for different particles.

Let V_1, V_2, V_3, ... be the linear velocities of the particles. Then the total K.E. of rotation of the body,

$$E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_3 v_3^2 + ...$$

$$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + ...$$

$$= \frac{1}{2} \omega^2 \left[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... \right]$$

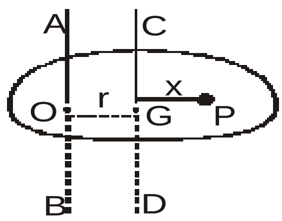
$= \frac{1}{2} \omega^2 I$, where I is the MI of the body about the axis rotation.

\ $\boxed{E = \frac{1}{2} I\omega^2}$

THEOREM OF PARALLEL AXES

Statement : The M.I of a body about any axis is equal to the sum of the M.I of the body about a parallel axis passing through its C.G. and the plroduct of the mass of the body and square of the distance b/w the two axes. This is called Steiner's theorem.

Proof : Consider a body of mass M whose C.G. is at G. Let the body rotate about an axis AOB and its M.I about about AOB be I. Let CGD be a parallel axis passing through G. Let the separation b/w the two axes be 'r'. Consider a particle of mass 'm' at P at a distance x from CGD. Then M.I of this particle about AOB = $m(r + x)^2$



M.I of the whole body about AOB is

$I = \Sigma m(r + x)^2$

$= \Sigma mr2 + \Sigma mx^2 + \Sigma 2mrx$

$= Mr^2 + I_G + 2 r \Sigma mx$.....(1), where

$\Sigma m = M$ is the total mass of the body,

$\Sigma mx^2 = I_G$ is the M.I of the body about CGD.

The weight of the particle of mass m at P = force acting on m = mg. The moment of this force about CGD = mgx.

$\therefore$ Sum of the moments of the weights of all the particles about CGD = Σmgx. The algebraic sum of the moments of all the forces about an axis passing through the centre of gravity of a body = 0.

\ $\Sigma mgx = 0$

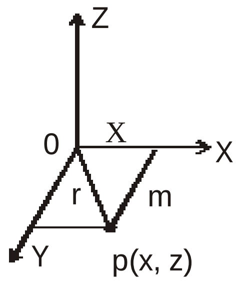
$\because g \neq 0, \Sigma mx = 0$

\ $2r \Sigma mx = 0$-----(2)

Substituting (2) in (1),

$\boxed{I = I_G + Mr^2}$ Hence the theorem

THEOREM OF PERPENDICULAR AXES



Statement : The M.I of a lamina about any axis $\perp^r$ to its surface is equal to the sum of the moments of inertia about two $\perp^r$ axes in the plane of the lamina, all the three axes passing through the same point on the lamina.

Let OX & OY be two rectangular axes in the plane of the lamina and OZ, an axis through `O' $\perp^r$ to both OX & OY.

Consider a particle of mass m at a point ρ distant `r' from O in the X - Y plane. Let its coordinates be x & y. Then $r^2 = x^2 + y^2$.

M.I. of the particle about z-axis = mr^2.

∴ M.I of the lamina about z-axis, $I_Z = \Sigma mr^2$.

$$\Rightarrow \quad I_Z = \Sigma m(x^2 + y^2)$$

Σmx^2 = M.I of the lamina about y-axis = I_y

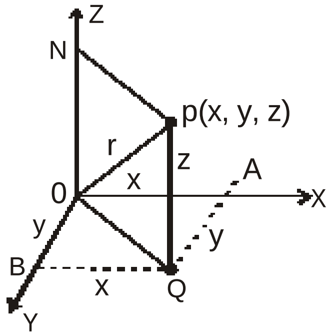
Σmy^2 = M.I of the lamina about x-axis = I_x

∴ $\boxed{I_Z = I_Y + I_X}$. Hence the theorem.

THEOREM OF PERPENDICULAR AXES IN THREE DIMENSIONS

Let OX, OY & OZ be three rectangular axes with the origin at O in the given body. Consider a particle of mass m at a point P. Let OP = r.

From P draw PQ ⊥r to the XY plane. Draw QB parallel to OX & QA parallel to OY. Then OA = x is the x-coordinate and OB = y is the y-coordinate of P. Join OQ, OQ is ⊥r to OZ. Draw PN parallel to OQ. Then PN is also ⊥r to OZ. ON = PQ = z is the Z-coordinate of P. and $r^2 = x^2 + y^2 + z^2$ ----- (1)



M.I of the particle at P about the Z-axis = $m \times NP^2$

$$= m \times NP^2 = m \times OQ^2 = m(OA^2 + OB^2)$$

$$= m(x^2 + y^2)$$

∴ M.I of the body about the Z-axis, $I_Z = \Sigma m(x^2 + y^2)$

$$\Rightarrow I_Z = \Sigma mx^2 + \Sigma my^2 \text{ -----(2)}$$

|||ly $I_Y = \Sigma mx^2 + \Sigma mz^2$ -----(3)

and $I_Z = \Sigma my^2 + \Sigma mz^2$ -----(4)

$$I_X + I_Y + I_Z = \Sigma my^2 + \Sigma mz^2 + \Sigma mx^2 + \Sigma mz^2 + \Sigma mx^2 + \Sigma my^2$$

$$= 2\Sigma mx^2 + 2\Sigma my^2 + 2\Sigma mz^2$$

$$= 2\Sigma m(x^2 + y^2 + z^2)$$

$\Rightarrow I_x + I_y + I_z = 2\Sigma mr^2$ {from (1)

$\Rightarrow I_x + I_y + I_z = 2\,I$

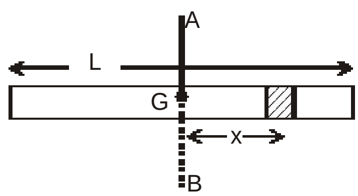
or $I = \frac{1}{2}(I_x + I_y + I_z)$

Thus the M.I of a three dimensional body about any axis passing through a point in the body is equal to half the sum of the M.I of the body about three mutually ⊥r axes passing through the same point.

1a) M.I of a thin rod about an axis perpendicular to its length and passing through its C.G.

Consider a thin uniform rod XY of length l and mass M, then its mass per unit length $m = \frac{M}{l}$. Let BGB be an axis ⊥r to its length and passing through its C.G.`G'. Consider a small element of thickness dx at a distance x from the axis of rotation. Then mass of the element = mdx. M.I of this element about AGB = $mdx \times x^2$

$$= mx^2\,dx$$



∴ M.I of the rod about AGB is

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} mx^2dx = m\left(\frac{x^3}{3}\right)_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{m}{3}\left(\frac{l^3}{8}+\frac{l^3}{8}\right) = \frac{ml^3}{12} = \frac{ml \cdot l^2}{12} = \frac{Ml^2}{12}$$

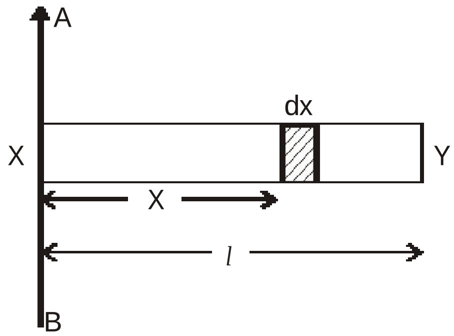
If K is the rodius of gyration of the rod about the axis of rotation, then $I = MK^2$

$$\therefore \quad MK^2 = \frac{Ml^2}{12} \Rightarrow K = \frac{l}{\sqrt{12}} = \frac{l}{2\sqrt{3}}$$

b) M.I of the rod about an axis at one end of the rod and perpendicular to the rod

Let AXB be an axis at one end and $\perp^r$ to the length of the rod.

Then $I = \int_0^l mx^2dx = \left(\frac{mx^3}{3}\right)_0^l = \frac{ml^3}{3} = \frac{ml \times l^2}{3}$



$\therefore \quad I = \frac{Ml^2}{3}$.

We know $I = MK^2$.

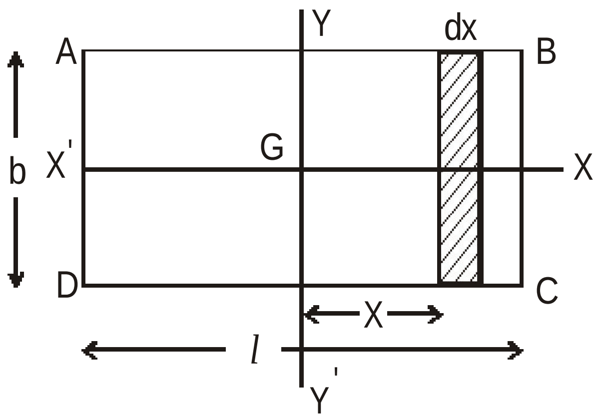
$\therefore \quad MK^2 = \frac{Ml^2}{3} \Rightarrow K = \frac{l}{\sqrt{3}}$

2. M.I of a rectangular lamina

(a) About an axis passing through its centre and parallel to breadth

Consider a rectangular lamina ABCD of mass M, length l and breadth b. Let m be its mass per unit area

i.e., $m = \frac{M}{l \times b}$, or $M = m(l \times b)$.



YY' is an axis ||el to its breadth. Consider a small strip of thickness dx at a distance x from the axis.

Its mass = m × (b × dx)

M.I of the strip about YY' = mbdx × x^2

= mbx^2 dx

$\therefore$ M.I of the rectangular lamina about YY' is

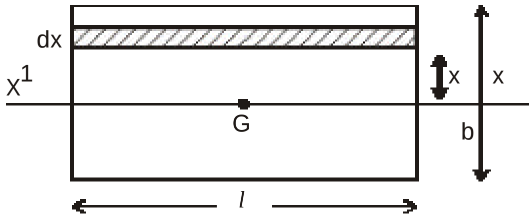
$$I_Y = \int_{-\frac{l}{2}}^{\frac{l}{2}} mbx^2\,dx = \frac{mb}{3}x^3\Big]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{mb}{3}\left(\frac{l^3}{8}+\frac{l^3}{8}\right) = \frac{mb}{3}\times\frac{2l^3}{8} = \frac{m(l\times b)\times l^2}{12} = \frac{Ml^2}{12}$$

$$\boxed{I_Y = \frac{Ml^2}{12}}$$

b) **Axis Passing through its Centre and parallel to length :**

$XX^{/}$ is an axis passing through its centre and $||$el to the length. Consider a small strip of thickness dx at a distance x from the axis.



Its mass = m × (l × dx).

M.I of the strip about $XX^{/}$ = m l d x × x^2

= m l x^2 dx

∴ M.I of the rectangular lamina about $XX^{/}$ is

$$I_X = \int_{-\frac{b}{2}}^{\frac{b}{2}} m\,l\,x^2\,dx$$

$$\Rightarrow I_x = \frac{ml}{3}x^3 = \frac{ml}{3}\times\frac{2b^3}{8} = \frac{m(l\times b)\times b^2}{12}$$

$$\Rightarrow I_X = \frac{ml}{3}x^3\Big]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{ml}{3}\times\frac{2b^3}{8} = \frac{m(l\times b)\times b^2}{12} = \frac{Mb^2}{12}$$

$$\Rightarrow \boxed{I_X = \frac{mb^2}{12}}$$

c) Axis Perpendicular to the plane of the lamina and passing through G

According to the theorem of $\perp^r$ axes, M.I of the lamina about an axis $\perp^r$ to the plane of the lamina and passing through G = M.I about two mutually $\perp^r$ axes through G in the plane of the lamina. Let I be the M.I of the lamina about an axis $ZGZ^{/}$ passing through its CG and $\perp^r$ to its plane.

i.e., $I = I_X + I_Y = \dfrac{Mb^2}{12} + \dfrac{Ml^2}{12}$

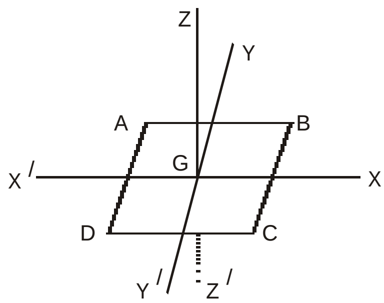
$$\Rightarrow I = \frac{M(l^2 + b^2)}{12}$$

(d) Axis along one end (AD)

$$I = \int_0^l m\,b\,x^2\,dx = \frac{mb}{3}l^3 = \frac{(mbl)l^2}{3}$$

$$\boxed{I = \frac{Ml^2}{3}}$$

M.I about AB : $I = \int_0^b m\,l\,x^2\,dx$



$$= \frac{mb}{3}\,b^3 = \frac{mlb}{3}b^2$$

$$\Rightarrow \boxed{I = \frac{Mb^2}{3}}$$

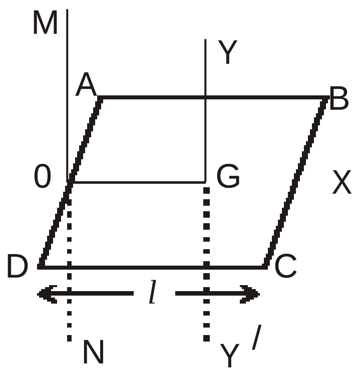
e) Axis passing through the mid point of AD or BC and perpendicular to the plane of the lamina

ABCD is the lamina. Let 0 be the mid point of AD; let MN be an axis through 0 ⊥r to the plane of the lamina. Let M.I of the lamina about MN be I, $YGY^/$ is a ||el axis through the centre of the lamina and to its plane. Let the M.I of the lamina about $Y0Y^/$ be I_G.

Then $I_G = \frac{M(l^2+b^2)}{12}$

By ||el axes theorem,

$$I = I_G + M\left(\frac{l}{2}\right)^2$$



$$\Rightarrow I = \frac{M(l^2+b^2)}{12} + M\left(\frac{l}{2}\right)^2$$

$$= \frac{Ml^2}{12} + \frac{Mb^2}{12} + \frac{Ml^2}{4}$$

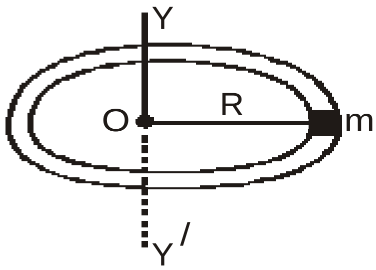
$$\boxed{I = \frac{Ml^2}{3} + \frac{Mb^2}{12}}$$

||| ly M.I about an axis through the mid point of AB or CD and ⊥r to the plane of the lamina

$$\boxed{I = M\left[\frac{b^2}{3} + \frac{l^2}{12}\right]}$$

3. **<u>M.I of a circular ring about an axis a) through its centre and perpendicular to its plane</u>**

Consider a thin circular ring of mass M and radius R. Let $Y0Y^/$ be the axis through its centre and ⊥r to its plane. Consider a small element of mass m. The M.I of this element about $Y0Y^/ = mR^2$.



∴ M.I of the ring about $Y0Y^/$, $I = \Sigma mR^2$.

$$\Rightarrow I = MR^2$$

∵ $\Sigma m = M$ and

R is the distance of each of the elements from 0

b) **<u>M.I of the ring about an axis along its diameter</u>**

Let $X0X^/$ and $Y0Y^/$ be two axes along two diameters of the ring. Then $I_X = I_Y$

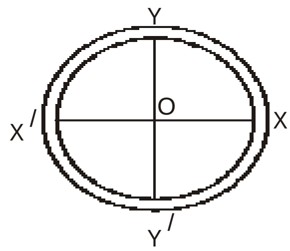
M.I of the ring is the same about all diameters. Let I_X and I_Y be the M.I of the ring about $X0X^/$ and $Y0Y^/$ respectively. Then $I_X = I_Y$.

Let the M.I of the ring about an axis through 0 and ⊥r to its plane be I. Then $I = MR^2$ where M is the mass and R is the radius of the ring.

By the theorem of $\perp^r$ axes, $I = I_X + I_Y$

$$\Rightarrow I = 2I_X = 2I_Y .$$

$$\Rightarrow I_x = I_y = \frac{I}{2} = \frac{MR^2}{2} .$$



∴ M.I of the ring about its diameter $= \dfrac{MR^2}{2}$

4) **M.I of a circulr lamina (disc) about an axis passing through its centre and perpendicular to its plane.**

Let $Y0Y^{/}$ be an axis through its centre and $\perp^r$ to its plane.

Consider a circular lamina of mass M and radius R.

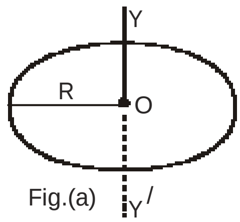


Fig.(a)

Area of the lamina $= \pi R^2$.

Mass per unit area of the lamina, $m = \dfrac{M}{\pi R^2}$.

The disc can be considered to be made up of a number of annular rings.

Consider one such ring of thickness dx and radius x.

Area of the ring $= 2\pi x\, dx$

Mass of the ring $= 2\pi x\, dx\, m$

$= 2\pi mx\, dx.$

M.I of the ring about $Y0Y^{/}$ $= 2\pi m x\, dx \times x^2$

$= 2\pi m x^3\, dx$

M.I of the lamina about $Y0Y^{/}$ $I = \int_0^R 2\pi m x^3\, dx$ is

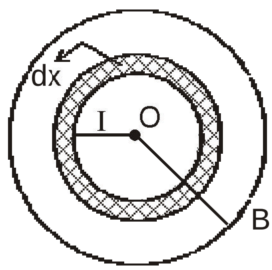


Fig.(b)

$$\Rightarrow I = 2\pi m \left[\frac{x^4}{4}\right]_0^R = \frac{\pi m R^4}{2}$$

$$= \frac{(\pi R^2 m)R^2}{2} \quad \Rightarrow \boxed{I = \frac{MR^2}{2}}$$

b) **M.I about its diameter**

Let $X0X^{/}$ and $Y0Y^{/}$ be two $\perp^r$ axes along two $\perp^r$ diameters of the disc.

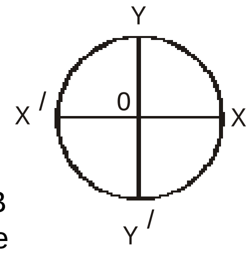
Let I_X and I_Y be the M.Is of the disc about $X0X^{/}$ and $Y0Y^{/}$ respectively

Then $I_X = I_Y$. M.I of the disc about an axis passing through its centre and $\perp^r$ to its plane, $I = \dfrac{MR^2}{2}$

By the theorem of $\perp^r$ axes, $I = I_X + I_Y = 2I_X = 2I_Y$

$$\Rightarrow \quad \frac{MR^2}{2} = 2I_X = 2I_Y .$$

\ $I_X = I_Y = \dfrac{MR^2}{4}$

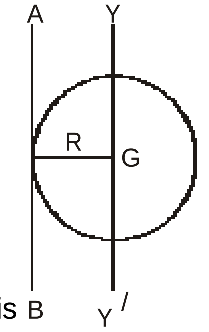


c) **<u>M.I. about a tangent</u>**

Let AB be an axis along a tangent to the disc. Let the M.I of the disc about AB be I. Let YGY$^/$ be an axis parallel to AB and along the diameter (through the C.G 'G'). Let the M.I of the disc about YGY$^/$ be I_G. Then $I_G = \dfrac{MR^2}{4}$, where M is the mass and R is the radius of the disc. By the theorem of parallel axes,

$I = I_G + MR^2$

$= \dfrac{MR^2}{4} + MR^2$



$\Rightarrow \boxed{I = \dfrac{5MR^2}{4}}$

5) **<u>M.I of a thin sphirical shell about diameter</u>**

Consider a thin spherical shell of a mass M and radius R. Let Y0Y$^/$ be an axis along a diameter of the shell.

Area of the shell = $4\pi R^2$.

\ Mass per unit area, $m = \dfrac{M}{4\pi R^2}$

The shell may be considered to be made up of a number of rings.

Let PQSR be one such ring of thickness dx and radius y at a distance x from Y0Y$^/$.

Area of the ring = $2\pi y\, dx$

From the figure, $\sin\theta = \dfrac{y}{R} \Rightarrow y = R\sin\theta$.

$\cos\theta = \dfrac{x}{R} \Rightarrow x = R\cos\theta$; $dx = -R\sin\theta\, d\theta$.

PR = dx = R dθ and $R^2 = x^2 + y^2 \Rightarrow y^2 = R^2 - x^2$.

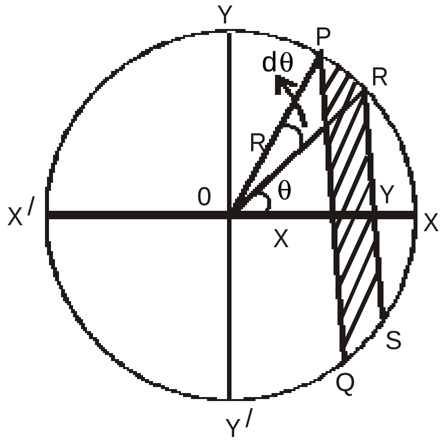
∴ Area of the ring $= 2\pi (R\sin\theta)(R\, d\theta)$

$= 2\pi R (R\sin\theta\, d\theta)$

$= 2\pi R\, dx$ (in magnitude)

∴ Mass of the ring $= 2\pi R\, dx\, m$

$= 2\pi m R\, dx$



M.I of the ring about X0X$^/$ (diameter)

$= \text{Mass} \times (\text{radius})^2$

$= 2\pi m R\, dx\, y^2$

$= 2\pi m R\, dx\, (R^2 - x^2)$.

M.I of the shell about X0X$^/$ is

$$I = \int_{-R}^{R} 2\pi m R (R^2 - x^2)\, dx$$

$$= \int_{-R}^{R} 2\pi m R^3 \, dx \; - \int_{-R}^{R} 2\pi m Rx^2 \, dx$$

$$= 2\pi m R^3 [2 R] - \frac{2\pi m R}{3}[2 R^3]$$

$$= 4\pi m R^4 - \frac{4\pi m R^4}{3}$$

$$= \frac{2 \times 4 \pi m R^4}{3} = \frac{4\pi R^2 m \times 2 R^2}{3}$$

$$\Rightarrow \boxed{I = \frac{2}{3} MR^2}$$

b) **M.I of a thin spherical shell about a tangent**

Let M.I of the shell about a tangent be I and about a ||el axis along its diameter (i.e., through its centre of gravity be I_G.)

Then by the theorem of parallel axes,

$I = I_G + MR^2$, where M is the mass of shell and R is its radius.

$$= \frac{2}{3} MR^2 + MR^2$$

$$\Rightarrow \boxed{I = \frac{5}{3} MR^2}$$

a) **M.I. of a solid sphere about its diameter**

Consider a solid sphere of radius R and mass M. Let ρ be the density of the material of the sphere.

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass of the sphere $= \frac{4}{3}\pi R^3 \rho$

The solid sphere may be considered to be made up of a number of concentric spherical shells of different radii. Consider one such shell of radius x and thickness dx.

Surface area of the shell $= 4 \pi x^2$.

Volume of the shell $= 4 \pi x^2 \, dx$

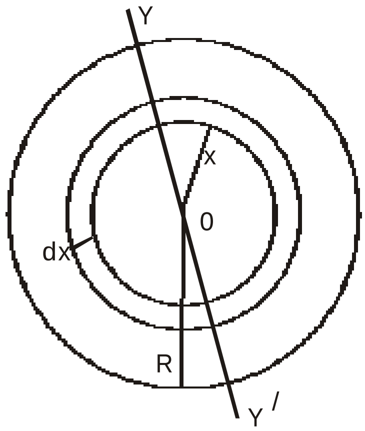
Mass of the shell, $=$ vol. $\times$ density

$= 4 \pi x^2 \, dx \, \rho$

M.I of the shell about $Y0Y'$ $= \frac{2}{3}$ Mass $\times$ (radius)^2

$$= \frac{2}{3} \, 4 \pi x^2 \, dx \; \rho \, x^2$$

$$= \frac{8}{3}\pi\rho x^4\, dx$$



M.I of the sphere about $Y0Y'$, $I = \int_0^R \frac{8}{3}\pi\rho x^4\, dx$

$$= \frac{8\pi\rho}{3\times 5}\left[R^5 - 0\right] = \left[\frac{4}{3}\pi R^3\rho\right] \times \frac{2}{5} R^2$$

$$\Rightarrow \boxed{I = \frac{2}{5} MR^2}$$

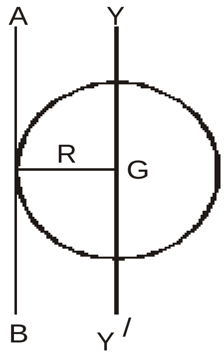
b) **M.I about a tangent**

By parallel axes theorem,

M.I about the tangent, $I = I_G + MR^2$

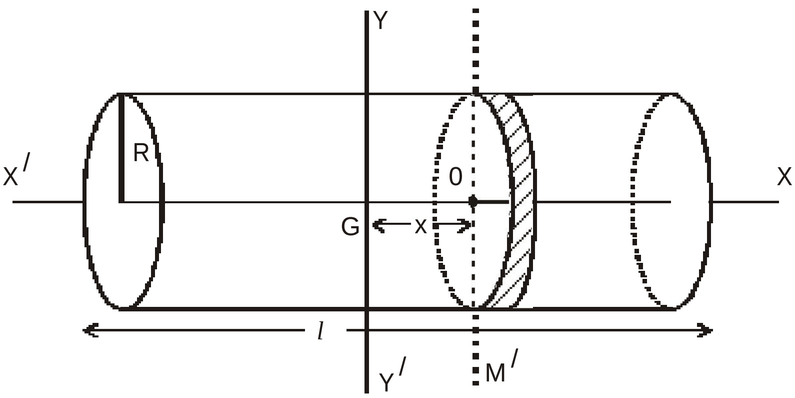
$$= \frac{2}{5} MR^2 + MR^2$$

$$\boxed{I = \frac{7}{5} MR^2}$$



M.I. of a solid cylinder about an axis passing through its centre and $\wedge^r$ to its length

Consider a solid cylinder of mass M, radius R and length l. Let YGY' be an axis passing through its centre of gravity and $\wedge^r$ to its length. Let m be the mass per unit length of the cylinder i.e., $m = \frac{M}{l}$.



The cylinder may be considered to be made up of a number of thin discs of rod R.

Consider one such disc of thickness dx at a distance x from YGY'. Mass of the disc = $m\,dx = \frac{M}{l} dx$

M.I of the disc about an axis $M0M'$ along its own diameter

$$= \frac{\text{Mass} \times (r \times d)^2}{4} = \frac{M}{l} dx \frac{R^2}{4} = \frac{MR^2}{4l} dx$$

M.I of the disc about YGY' = M.I of the disc about $M0M'$ + Mass × x^2 (by ||el axes theorem)

$$= \frac{MR^2}{4l}\,dx + \frac{M}{l}\,dx\,x^2$$

$$= \frac{MR^2}{4l}\,dx + \frac{M}{l}\,x^2\,dx$$

$$= \frac{M}{l}\left(\frac{R^2}{4}\,dx + x^2\,dx\right)$$

M.I of the cylinder about YGY$^{/}$ is given by

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{M}{l}\left(\frac{R^2}{4}\,dx + x^2\,dx\right)$$

$$= \frac{M}{l}\left.\frac{R^4}{4}\right]_{-l/2}^{l/2} + \frac{M}{l}\left.\frac{x^3}{3}\right]_{-l/2}^{l/2}$$

$$= \frac{MR^4}{4} + \frac{M}{3\,l}\left(\frac{l^3}{8} + \frac{l^3}{8}\right)$$

$$= \frac{MR^2}{4} + \frac{M\,l^3}{12\,l}$$

$$= \frac{MR^2}{4} + \frac{M\,l^3}{12}$$

$$\Rightarrow \quad \boxed{I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)}$$

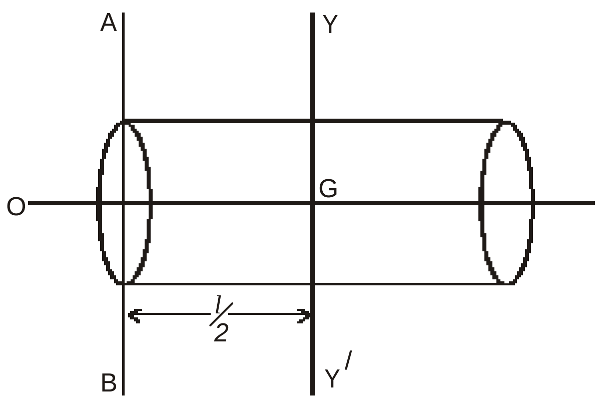
b) **<u>M.I about the diameter of one of its faces</u>**

$$I = \text{M.I about YGY}^{/} + \text{Mass} \times \left(\frac{l}{2}\right)^2$$

$$\Rightarrow \quad I = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right) + \frac{M\,l^2}{4}$$

$$= M\left(\frac{R^2}{4} + \frac{l^2 + 3\,l^2}{12}\right)$$

$$\boxed{I = M\left(\frac{R^2}{4} + \frac{l^2}{3}\right)}$$



c) **M.I about its own axis**

M.I of the disc about an axis pasisng through its centre and $\perp^r$ to its plane =

M.I of the disc about the axis $XOX^/$ of the cylinder = $\frac{\text{Mass of the disc} \times \text{rad}^2}{2}$

$$= \frac{MR^2}{2}$$

M.I of the cylinder about $XOX^/$ is

$$I = \sum \frac{mR^2}{2} \Rightarrow I = (\Sigma m)\frac{R^2}{2}$$

$$\Rightarrow = \boxed{I = \frac{MR^2}{2}}$$