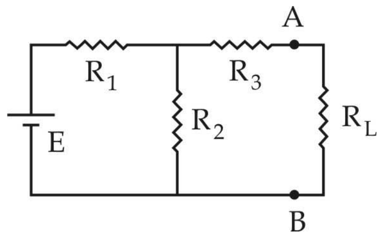
NETWORK THEOREMS

1. Thevenin's theorem

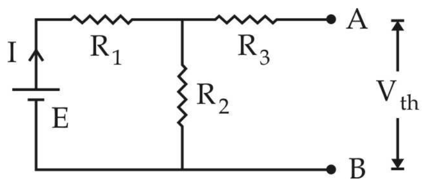
Statement: *A linear network consisting of a number of voltage sources and resistances can be replaced by an equivalent network having a single voltage source called Thevenin's voltage* (V_{Th}) *and a single resistance called Thevenin's resistance* (R_{Th}).

Explanation:



Consider a network or a circuit as shown. Let E be the emf of the cell having its internal resistance $r=0$. $R_L \to$ load resistance across AB.

To find V_{Th}:

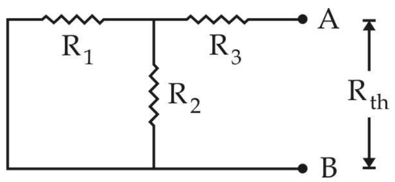


The load resistance R_L is removed. The current I in the circuit is $I=\dfrac{E}{R_1+R_2}$.

The voltage across $AB=$ Thevenin's voltage V_{Th}.

$$V_{Th}=I\,R_2 \quad \Rightarrow \quad \boxed{V_{Th}=\frac{E\,R_2}{R_1+R_2}}$$

To find R_{Th}:



The load resistance R_L is removed. The cell is disconnected and the wires are short as shown.

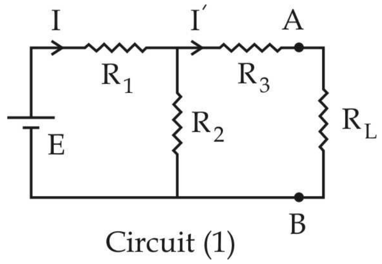
The effective resistance across $AB=$ Thevenin's resistance R_{Th}.

$\boxed{R_{Th}=R_3+\dfrac{R_1R_2}{R_1+R_2}}$ [R_1 is parallel to R_2 and this combination in series with R_3]

If the cell has internal resistance r, then $V_{Th}=\dfrac{E\,R_2}{R_1+R_2+r}$ and $R_{Th}=R_3+\dfrac{(R_1+r)R_2}{R_1+r+R_2}$.

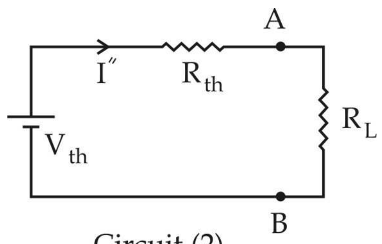
Proof of Thevenin's theorem:

Consider the network as shown below



Circuit (1)

The equivalent circuit is given by



Circuit (2)

The effective resistance of the network in (1) is R_3 and R_L in series and this combination is parallel to R_2 which in turn is in series with R_1.

Thus, $R_{eff} = R_1 + \dfrac{R_2(R_3+R_L)}{R_2+R_3+R_L}$ -------------- (1)

The current I in the circuit is $I = \dfrac{E}{R_{eff}} = \dfrac{E}{R_1 + \dfrac{R_2(R_3+R_L)}{R_2+R_3+R_L}}$

or $I = \dfrac{E(R_2+R_3+R_L)}{R_1R_2+R_1R_3+R_1R_L+R_2R_3+R_2R_L}$ ----------- (2)

The current through the load resistance (I') is found using branch current method.

$I' = \dfrac{I\,R_2}{R_2+R_3+R_L}$ ----------- (3)

Substituting for I from (2) in (3)

$$I' = \frac{E(R_2+R_3+R_L)R_2}{(R_2+R_3+R_L)(R_1R_2+R_1R_3+R_1R_L+R_2R_3+R_2R_L)}$$

or $I' = \dfrac{ER_2}{R_1R_2+R_1R_3+R_1R_L+R_2R_3+R_2R_L}$ ------------ (4)

Thevenin's voltage $V_{Th} = \dfrac{ER_2}{R_1+R_2}$ ----------- (5)

Thevenin's resistance $R_{Th} = R_3 + \dfrac{R_1R_2}{R_1+R_2}$ ---------- (6)

Consider the equivalent circuit (circuit (2))

The current I'' in the equivalent circuit is $I'' = \dfrac{V_{Th}}{R_{Th}+R_L}$ ----------- (7)

Substituting for V_{Th} and R_{Th} from (5) and (6) in (7)

$$I'' = \frac{E R_2}{R_1+R_2} \times \frac{1}{R_3 + \dfrac{R_1 R_2}{R_1+R_2} + R_L} = \frac{E R_2}{(R_1+R_2)\dfrac{R_3R_1 + R_3R_2 + R_1R_L + R_2R_L + R_1R_2}{(R_1+R_2)}}$$

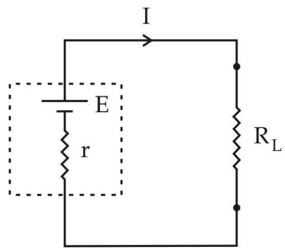
or $$I'' = \frac{E R_2}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L} \text{ ------------- (8)}$$

From equations (4) and (8), it is observed that $I' = I''$.

Hence Thevenin's theorem is verified.

2. Maximum Power Transfer Theorem

Statement: *The power transferred by a source to the load resistance in a network is maximum when the load resistance is equal to the internal resistance of the source.*



Proof of Maximum power transfer theorem:

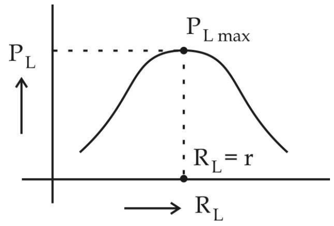
Consider a network with a source of emf E and internal resistance r connected to a load resistance R_L. The current I in the circuit is

$$I = \frac{E}{R_L + r} \text{ ----------- (1)}$$

The power delivered to load resistance R_L is $P_L = I^2 R_L$ or $P_L = \left(\dfrac{E}{R_L + r}\right)^2 R_L$

(from equation (1))

$$P_L = \frac{E^2}{(R_L + r)^2} R_L \text{ ---------- (2)}$$



The variation of P_L with R_L is as shown.

P_L is found to be maximum for a particular value of R_L when

P_L is maximum, $\dfrac{dP_L}{dR_L} = 0$

[$\because$ No variation of P_L with R_L at $P_{L\,\max}$]

i.e., $\dfrac{d}{dR_L}\left(\dfrac{E^2 R_L}{(R_L + r)^2}\right) = 0$ or $\dfrac{d}{dR_L}\left[E^2 R_L (R_L + r)^{-2}\right] = 0$

Differentiating $E^2\left[R_L(-2)(R_L + r)^{-3} + (R_L + r)^{-2}\right] = 0$ or $\dfrac{-2R_L}{(R_L + r)^3} + \dfrac{1}{(R_L + r)^2} = 0$ or

$$\frac{2R_L}{(R_L + r)^3} = \frac{1}{(R_L + r)^2}$$

Thus, $\dfrac{2R_L}{R_L + r} = 1 \quad \Rightarrow \quad 2R_L = R_L + r$ or $\boxed{R_L = r}$

Thus the power delivered to the load resistance is maximum when the load resistance is equal to the internal resistance of the source.

To show that the maximum power transfer efficiency of a circuit is 50%:

The power across the load $P_L = I^2 R_L = \frac{E^2}{(R_L + r)^2} R_L$ ----------- (1)

From the maximum power transfer theorem, P_L is maximum when $R_L = r$. Putting this condition in equation (1),

$$P_{L\max} = \frac{E^2}{(2R_L)^2} R_L \Rightarrow P_{L\max} = \frac{E^2}{4R_L} \quad \text{------------- (2)}$$

The power that is taken from the voltage source is (or power generated by the source), $P = I^2 (R_L + r) = \frac{E^2}{(R_L + r)^2}(R_L + r)$ or $P = \frac{E^2}{R_L + r}$. When $R_L = r$, $P = \frac{E^2}{2R_L}$ -------- (3)

Dividing equation (2) by (3)

$$\frac{P_{L\max}}{P} = \frac{E^2}{4R_L} \times \frac{2R_L}{E^2} = \frac{1}{2} \text{ or } P_{L\max} = \frac{P}{2}$$

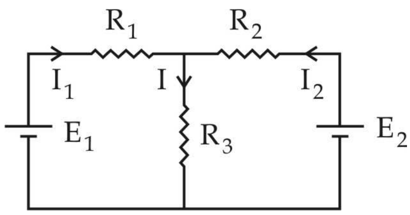
Thus the maximum power delivered to the load is only half the power generated by the source or the maximum power transfer efficiency is 50%. The remaining 50% power is lost across the internal resistance of the source.

3. Superposition theorem

Statement: *In a linear network having number of voltage or current sources and resistances, the current through any branch of the network is the algebraic sum of the currents due to each of the sources when acting independently.*

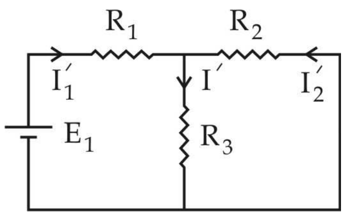
Explanation: By mesh current analysis.

1. Consider the network as shown. The currents in different branches of the network are I_1, I_2 and I as shown. Also $I_1 + I_2 = I$.

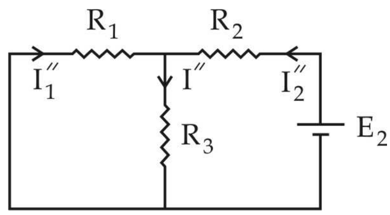


2. [Let the internal resistance r of the cells be negligible].

   The cell E_2 is removed and the terminals are short as shown. Now the currents in the branches are I_1', I_2' and I'. Also $I' = I_1' + I_2'$.



3. The E_1 is removed and the terminals are short as shown. The currents are I_1'', I_2'' and I''. Also $I'' = I_1'' + I_2''$.



According to superposition theorem $I_1 = I_1' + I_1''$. $I_2 = I_2' + I_2''$ and $I = I' + I''$

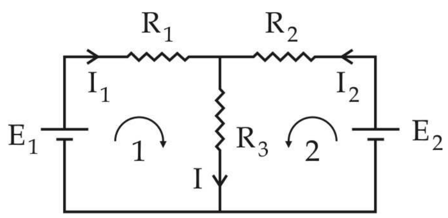
$\boxed{I = I_1 + I_2}$

Verification of superposition theorem:

1. Consider the network shown. Applying Kirchhoff's voltage to the loop 1.

$I_1R_1 + I_1R_3 + I_2R_3 = E_1$ $\quad [\because I_1R_1 + I(R_3) = E_1 \quad I = I_1 + I_2]$

or $I_1 = \dfrac{E_1 - I_2R_3}{R_1 + R_3}$ ----------- (1)



Considering loop 2, $I_2R_2 + IR_3 = E_2$.

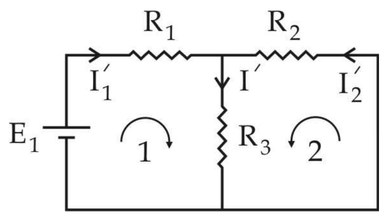
$I_2R_2 + I_1R_3 + I_2R_3 = E_2$

$I_2 = \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ---------- (2)

Thus, $I = I_1 + I_2$

$I = \dfrac{E_1 - I_2R_3}{R_1 + R_3} + \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ----------- (3)

2. Consider the circuit shown with E_2 removed and terminals short. Applying Kirchhoff's law to loop 1.



$I_1'R_1 + I'R_3 = E_1$ $\qquad$ As $\quad I' = I_1' + I_2'$,

$I_1R_1 + I_1'R_3 + I_2'R_3 = E_1 \Rightarrow \quad I_1' = \dfrac{E_1 - I_2'R_3}{R_1 + R_3}$ ---------- (4)

Similarly for loop 2,

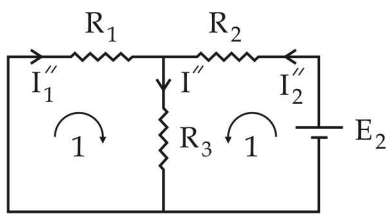
$I_2'R_2 + I'R_3 = 0 \quad \Rightarrow \qquad I_2'R_2 + I_1'R_3 + I_2'R_3 = 0$

$I_2' = -\dfrac{I_1'R_3}{R_2 + R_3}$ ----------- (5)

$I' = I_1' + I_2' = \dfrac{E_1 - I_2'R_3}{R_1 + R_3} - \dfrac{I_1'R_3}{R_2 + R_3}$ ....(6)

3. Consider the circuit with E_1 removed and terminals short.

For loop (1) $I_1''R_1 + I''R_3 = 0$



As $I'' = I_1'' + I_2''$

$$I_1''R_1 + I_1''R_3 + I_2''R_3 = 0 \quad \Rightarrow \quad I_1'' = \frac{-I_2''R_3}{R_1 + R_3} \text{ ------------ (7)}$$

For loop (2)

$$I_2''R_2 + I''R_3 = E_2 \quad \Rightarrow \quad I_2''R_2 + I_1''R_3 + I_2''R_3 = E_2$$

or $I_2'' = \dfrac{E_2 - I_1''R_3}{R_2 + R_3}$ ------------- (8)

$$I'' = I_1'' + I_2'' = \frac{-I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3} \text{ -------------- (9)}$$

Adding equations (6) and (9)

$$I' + I'' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3} - \frac{I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3}$$

$$= \frac{1}{R_1 + R_3}\left[E_1 - I_2'R_3 - I_2''R_3\right] + \frac{1}{R_2 + R_3}\left[E_2 - I_1''R_3 - I_1'R_3\right]$$

$$I' + I'' = \frac{1}{R_1 + R_3}\left[E_1 - R_3\left(I_2' + I_2''\right)\right] + \frac{1}{R_2 + R_3}\left[E_2 - R_3\left(I_1' + I_1''\right)\right] \text{ ....(10)}$$

Comparing equations (3) and (10) it is observed that

$I_1 = I_1' + I_1''$

$I_2 = I_2' + I_2''$

$I = I' + I''$

Hence the proof of the theorem.