DC CIRCUIT ANALYSIS

Syllabus:: Concept of Voltage and Current Sources, Kirchhoff's Current Law, Kirchhoff's Voltage Law (statements). Principle of Duality (voltage and current source equivalents). Thevenin's Theorem (statement and proof), Superposition Theorem(statement and proof), Norton's Theorem (Statement and explanation). Reciprocity Theorem. Maximum Power Transfer Theorem (statement and proof).

Electrical Circuit/Network is a combination of several circuit elements like resistors, capacitors and inductors and sources of emf/voltage connected in any manner. The circuit is said to be linear if the parameters like resistance, capacitance etc.. are constant which do not depend on the voltage or current.

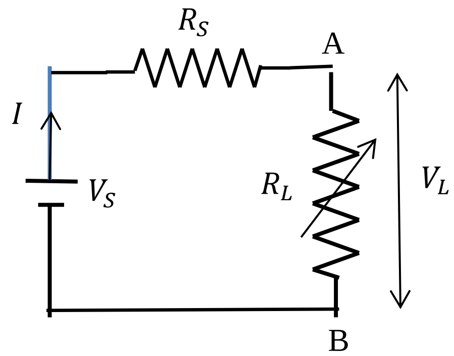
Electrical junction or node is a point in an electrical network where two or more conductors meet. **Loop or mesh** is a closed path for current in a network.

Ohm's law : The electric current through a circuit is directly proportional to the the applied potential difference (voltage) across its terminals provided the other physical conditions like temperature remains constant. Mathematically it is represented by $\boldsymbol{V = IR}$ **,** where V indicates the applied voltage, I is the current and R is the resistance offered to the flow of current.

Voltage Source: A device which can produce a continuous force to move the electrons through the wire connected into the two terminals of the device or a device which can produce continuous voltage is called a Voltage Source. It is the electrical energy required to drive the current. There are two types of the Voltage Sources.

1. A device which produces a continuous direct voltage output is called a Direct Voltage Source. For example: Cells , Battery , DC Generator. A direct voltage is the kind of voltage whose polarity remains the same. Direct Voltage causes the current to move only in one direction continuously.
2. A device which produces an alternating voltage output is called a Alternating Voltage Source. For example: AC Generator , DC to AC converter etc. A alternating voltage is the kind of voltage whose polarity is reversed periodically. Alternating Voltage causes the current to move in one direction during one half the cycle or half a period then in another direction in the next half cycle.

Concept of a voltage source :



Voltage source with load connected

Consider a circuit in which a voltage source in the form of a cell of voltage V_S and internal resistance R_S is connected in series with a load resistance R_L as shown in diagram.

The work is done by the voltage source in driving current through (1) the load resistance and (2) the internal resistance.

Thus if V and V' are the work done in driving current I through R_L and R_S respectively, then by the law of conservation of energy

V_S = V + V' ………(1)

The potential difference across $R_L = V = IR_L$ (from ohm's law)

The potential difference across R_S = V' = IR_S.

Substituting for V and V' in equation (I) $V_S = IR_L + IR_S = I(R_L + R_S)$

Or $I = \frac{V_S}{R_L + R_S}$ …(2)

The potential difference between the terminals of the voltage source when the source is in closed circuit is called the **terminal voltage of the voltage source**.

If V_L is the voltage across the load resistance R_L which is the potential difference across the terminals A and B called terminal voltage, then $V_L = IR_L$ …..(3)

Substituting for I from equation (2) in (3) we get $V_L = \left(\frac{V_S}{R_L + R_S}\right) R_L$

or $V_L = \left(\frac{V_S}{1 + \frac{R_S}{R_L}}\right)$ …(4)

The equation (4) is the expression for terminal voltage of the voltage source. If $\frac{R_S}{R_L}$ is small compared to one, the terminal voltage is same as the open circuit voltage or emf of the voltage source. Under this condition the source behaves as a good voltage source i.e. the terminal voltage remains constant even if the load resistance of the external resistance varies.

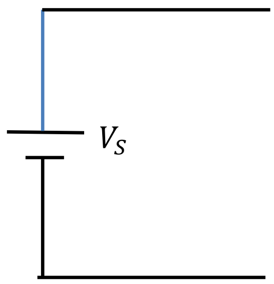
From equation (4) it is observed that terminal voltage $V_L = V_S$ when either $R_S = 0$ or $R_L = \infty$. In the first case $V_L = V_S$ when the internal resistance is zero which is possible in case of an ideal voltage source. This is practically not possible.

In the second case $V_L = V_S$ when $R_L = \infty$. This is the case of open circuit. Thus emf of the voltage source is equal to potential difference across its terminals when the voltage source is in **open circuit**.

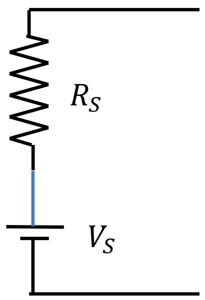
In general V_L is always less than V_S as given by the equation $V_L = V_S - IR_S$. This is the case of a practical voltage source.

<u>Ideal Voltage Source:</u> An Ideal Voltage source is a voltage source whose internal resistance is zero, such that the supplied voltage does not change even if the external load resistance changes. The internal resistance of a ideal voltage source is zero.

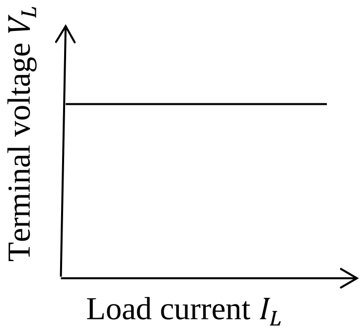
But practically no matter how much efforts are made the voltage source still have a small internal resistance. But a voltage source can be converted into a Virtually Constant Voltage Source by changing the internal materials used in a cell or voltage source such that the internal resistance of the source is minimized.
A practical virtual constant voltage source has a very low internal resistance and the actual circuit diagram of a voltage source is as shown. In general a voltage source is as shown in second diagram



Ideal Voltage source



Practical Voltage source



Variation in case of
Ideal Voltage source

The variation of terminal voltage with load current is as shown in the graph above in case of ideal voltage source.

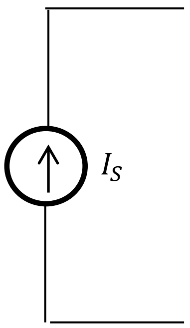
<u>Current Source:</u>
A current source is a device which provides the regular flow of electrons or current in a circuit. A current source is a type of voltage source which have enough EMF and surplus electrons so as to produce the flow of electrons.

Direct Current Source: The current source made of a Direct Voltage Source is called Direct Current Source.
Alternating Current Source: The current source made of a Alternating Voltage Source is called Alternating Current Source.

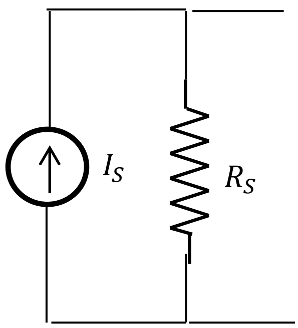
Ideal Current Source:

A current source which provides a constant current without any relation with the voltage supplied to the load is called Ideal Current Source. The current flowing through the load resistance is independent of the load resistance. i.e. the current does not vary even if the load resistance changes. The internal resistance of an ideal current source is infinite.

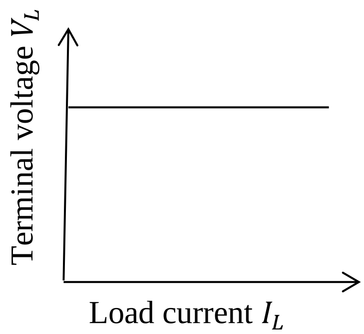
Practically an Ideal Current source is impossible but a circuit can be configured such that when the voltage across the load is changed the supplied current varies negligibly. A virtual Constant or Ideal current source can be made by adding a very high internal resistance to a Voltage Source as shown in the figure below:



Ideal current source



Practical current source

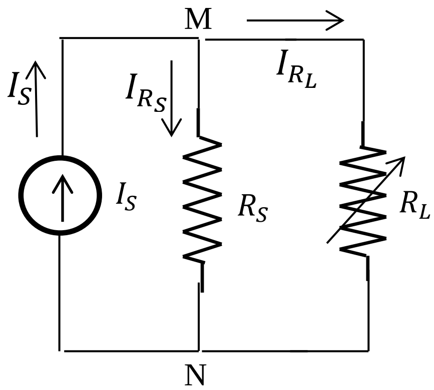


Variation in case of ideal current source

Consider a current source of current $\boldsymbol{I_S}$ and internal resistance $\boldsymbol{R_S}$ in parallel connected to a load resistance $\boldsymbol{R_L}$ as shown in the diagram. At M the current gets divided as shown. Thus $\boldsymbol{I_S} = \boldsymbol{I_{R_S}} + \boldsymbol{I_{R_L}}$. or $\boldsymbol{I_{R_S}} = \boldsymbol{I_S} - \boldsymbol{I_{R_L}}$ …(1)

As R_S and R_L are in parallel, the voltage across each of them is same.

Thus $\boldsymbol{I_{R_S}}\, R_S = \boldsymbol{I_{R_L}}\, R_L$ …..(2)



Practical current source

(Note: In a circuit with resistors in series the voltage gets divided and current is the same. The total voltage is the sum of the voltages across each resistor.

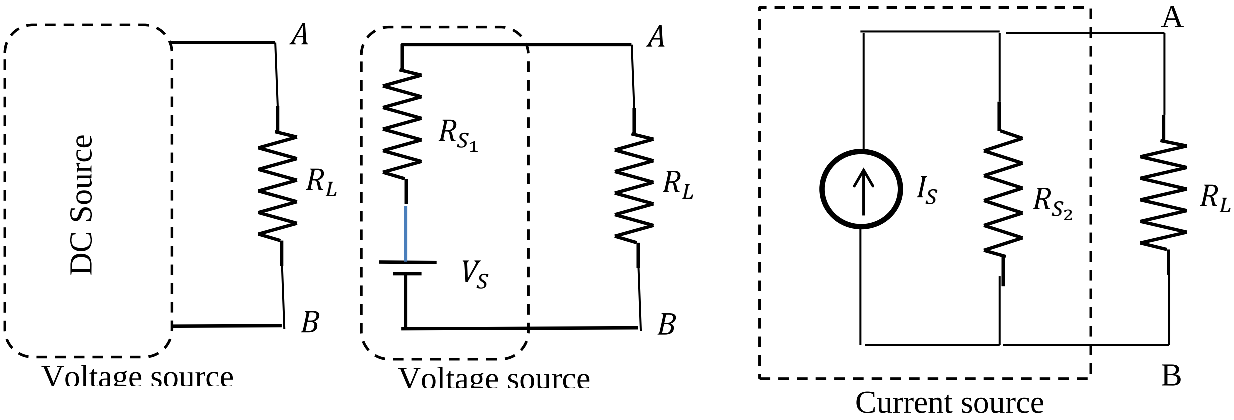
In a circuit with resistors in parallel, the current gets divided and the voltage remains the same. The total current is the sum of the currents through each of the resistors.)

Substituting for $\boldsymbol{I_{R_S}}$ from (1) in (2) $\left(\boldsymbol{I_S} - \boldsymbol{I_{R_L}}\right) R_S = \boldsymbol{I_{R_L}}\, R_L$

or $\boldsymbol{I_{R_L}}(R_S + R_L) = \boldsymbol{I_S}\, R_S$ $\quad \boldsymbol{I_{R_L}} = \frac{I_S}{R_S + R_L} R_S = \frac{I_S}{1 + \frac{R_L}{R_S}}$ .....(3)

From equation (3) it is observed that $\boldsymbol{I_{R_L}} = \boldsymbol{I_S}$ provided $\frac{R_L}{R_S}$ is small compared to one. Thus R_S should be very large compared to R_L. Under this condition, the current through the load will almost remain same even if the load resistance is varied. This is a practical current source.

Equivalence between voltage and current sources (Duality)



Consider a DC source which can be considered as voltage or a current source as shown. If the load resistance $\boldsymbol{R_L}$ is large compared to internal resistance $\boldsymbol{R_S}$ then the source is considered as a voltage source. If the load resistance $\boldsymbol{R_L}$ is small compared to internal resistance $\boldsymbol{R_S}$ then the source is considered as a current source

The equivalence between the two can be explained as follows.

The voltage source consists of an ideal voltage source V_S in series with the internal resistance R_{S_1}. The current source consists of an ideal current source I_S in parallel with the internal resistance R_{S_2} as shown. Both of the sources are from the same common source and hence must give same results. If in the voltage source circuit the value of R_L is made zero, then the A and B are short. Then the load current is $\boldsymbol{I_L} = \frac{\boldsymbol{V_S}}{\boldsymbol{R_{S_1}}}$ **.......(1)**

In the current source circuit, if $\boldsymbol{R_L}$ is made zero, then the load current is same as the source current $\boldsymbol{I_S}$.(This is because $\boldsymbol{R_S}$ and $\boldsymbol{R_L}$ are in parallel, when $\boldsymbol{R_L}$ is zero net resistance is zero and thus $\boldsymbol{I_L} = \boldsymbol{I_S}$)

Thus . $\boldsymbol{I_L} = \boldsymbol{I_S} = \frac{\boldsymbol{V_S}}{\boldsymbol{R_{S_1}}}$. ….(2) Thus both are equivalent.

2. Let the load resistance R_L be removed so that the circuit becomes open (i.e. $R_L = infinity$). In case of voltage source circuit, if R_L is removed, the circuit is open and the voltage across load is voltage across AB equal to the source voltage. i.e. $V_L = V_S$ …..(3).

In case of current source circuit, if $\boldsymbol{R_L}$ is removed then voltage across AB is

$V_L = I_S \boldsymbol{R_{S_2}}$ ….(4)

From (3) and (4) it is observed that the two circuits are equivalent as

$V_L = V_S = I_S \boldsymbol{R_{S_2}}$ **…(5)**

From equation (2) $.\boldsymbol{V_S} = \boldsymbol{I_S R_{S_1}}$ **….(6)**

Equations (5) and (6) show that $\boldsymbol{R_{S_1}} = \boldsymbol{R_{S_2}}$

If this source resistance is equal to $\boldsymbol{R_S}$, then in general $\boldsymbol{V_S} = \boldsymbol{I_S R_S}$. Thus the two circuits are equivalent.

3. If I_{L_1} is the current through the load resistance in voltage source circuit, then

$I_{L_1} = \frac{V_S}{R_S+R_L}$ …(7)

If $\boldsymbol{I_{L_2}}$ is the current through the load resistance in current source circuit then

$I_{L_2} = \boldsymbol{I_S} \frac{\boldsymbol{R_S}}{R_S+R_L}$ or $I_{L_2} = \frac{V_S}{R_S+R_L}$ ……(8) (As $\boldsymbol{V_S} = \boldsymbol{I_S R_S}$)

From (7) and (8) it is seen that $I_{L_1} = I_{L_2}$

The above relations show that the current through any value of load resistance is the same whether the circuit is voltage source circuit or current source circuit. Thus the equivalence between the two sources.

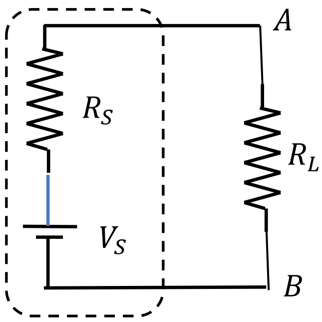
Conversion of Voltage Source into Current Source

Consider the voltage source circuit as shown. The load current through the load resistance in the circuit is given by $I_L = \frac{V_S}{R_S+ R_L}$ …..(1)

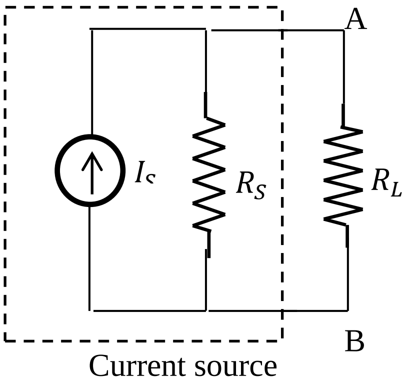
Multipling and dividing the above equation by R_S we have $I_L = \frac{V_S/R_S}{(R_S+ R_L)/R_S}$

The above equation van be written as $I_L = \frac{V_S}{R_S} \times \frac{R_S}{R_S + R_L}$

Or $I_L = I_S \times \frac{R_S}{R_S + R_L}$ ….(2) where $I_S = \frac{V_S}{R_S}$ is the current that would flow in the circuit if the output terminals are short circuited i.e. when load resistance is made zero. Equation (2) shows that the voltage source appears as the current source of value I_S which is divided between the internal resistance R_S and load resistance R_L connected in parallel. Thus a real voltage source of voltage V_S and internal resistance R_S is equivalent to the current source of value $I_S = \frac{V_S}{R_S}$ and R_S in parallel with the current source.



Voltage source



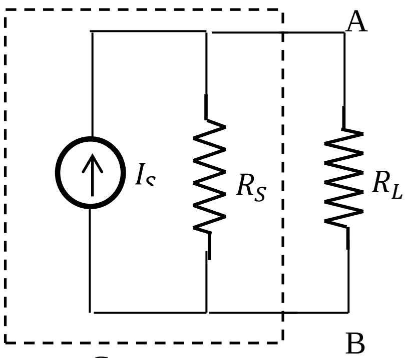
Current source

Conversion of Current Source into Voltage Source

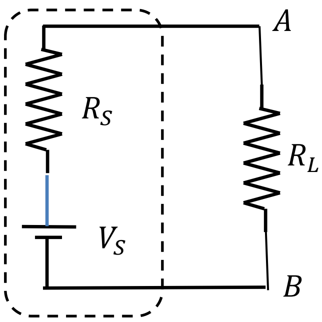
Consider the current source circuit as shown. The load current through the load resistance in the circuit is given by $I_L = I_S \times \frac{R_S}{R_S + R_L}$ …..(1)

The above equation van be written as $I_L = \frac{V_S}{R_S + R_L}$ (2)

where $V_S = I_S R_S$ is the voltage that appears across the output terminals AB if the circuit is open, i.e.load resistance is removed or load resistance is Equation (2) shows that the current source appears as the voltage source of value V_S which appears across the voltage source and the internal resistance in series along with the load resistance.



Current source



Voltage source

Thus a real current source of current I_S and internal resistance R_S is equivalent to the voltage source of value $V_S = I_S R_S$ and R_S in series with the voltage source.

(Note : Branch current : When two resistances are in parallel, the current through any any one branch is given by

$$Current\ in\ one\ branch = \frac{main\ current \times resistance\ in\ the\ other\ branch}{sum\ of\ the\ resistences}$$ **)**

Kirchhoff's laws

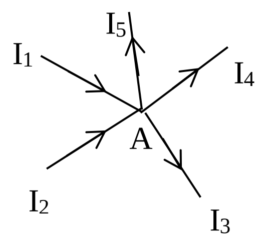
Robert Kirchhoff formulated two laws for the analysis of complex electrical circuits. These laws are known as Kirchhoff's current law(KCL) and Kirchhoff's voltage law (KVL).

I Law or Current Law

This is based on the law of conservation of charge

Statement In an electrical network the algebraic sum of the currents at a node is zero. i.e., $\Sigma I = 0$

Illustration



Let the currents I_1, I_2, I_3 and I_4 be the currents flowing in different branches at a junction *A* as shown in the diagram. By convention the current entering a node is taken to be positive and that leaving the node is taken to be negative. Hence I_1 and I_2 are positive while I_3, I_4 and I_5 are taken as negative.

According to Kirchhoff's current law $I_1 + I_2 - I_3 - I_4 - I_5 = 0$

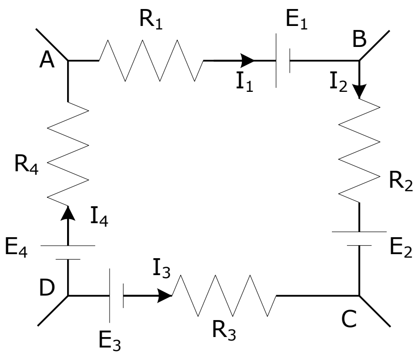
or $I_1 + I_2 = I_3 + I_4 + I_5$

Thus the sum of the currents entering a node is equal to sum of the currents leaving it.

II Law or Voltage Law: In a current loop of an electrical network the algebraic sum of the products of current and resistances in different loops is equal to the algebraic sum of the emfs of the cells in that loop. i.e. $\Sigma IR = \Sigma E$

or the algebraic sum ot the voltages across the resistors and sources of voltages in a loop is zero i.e. $\Sigma V = 0$.

Illustration : Consider a loop ABCDA consisting of cells and resistors in a network as shown. The directions of currents, voltages across the resistors and the emf's of cells are as shown. The following conventions are used to apply the voltage law.



(1) In resistors potential falls in the direction of flow of conventional current. i.e., (IR) product is taken as negative. Otherwise it is taken as positive.

(2) In cells potential falls while traversing from positive to negative terminal. i.e., emf is - E. Potential rises while traversing from negative to positive terminal. i.e., emf is + E.

Appling KVL to the network shown, we get

$- I_1R_1 - E_1 - I_2R_2 - E_2 + I_3R_3 + E_3 - I_4R_4 + E_4 = 0$

or $- I_1R_1 - I_2R_2 + I_3R_3 - I_4R_4 = E_1 + E_2 - E_3 - E_4$

The Kirchhoff laws are applicable to both AC and DC circuits.

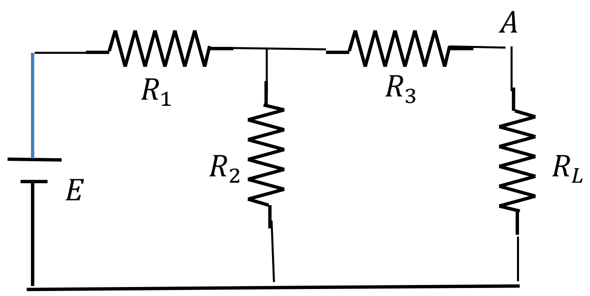
Network Theorems

1. Thevenin's theorem

Statement: A linear network consisting of a number of voltage sources and resistances can be replaced by an equivalent network having a single voltage source called Thevenin's voltage (V_{Th}) and a single resistance called Thevenin's resistance (R_{Th}).

Explanation:

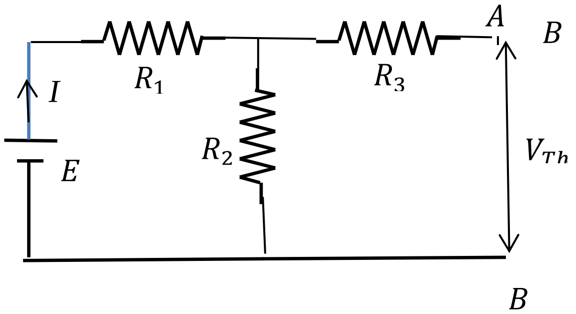
Consider a network or a circuit as shown. Let E be the emf of the cell having its internal resistance $r = 0$. Let R_L be the load resistance across AB.



To find V_{Th}:

The load resistance R_L is removed. The current I in the circuit is $I = \frac{E}{R_1 + R_2}$.

The voltage across $AB =$ Thevenin's voltage V_{Th}.

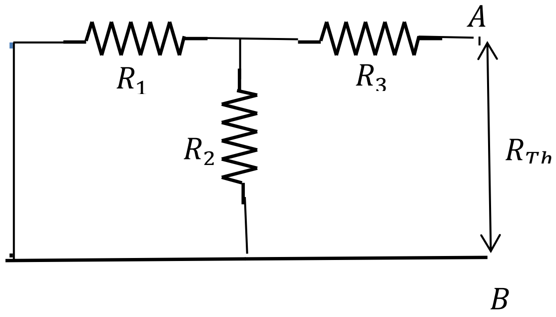


$$V_{Th} = I R_2 \quad \Rightarrow \quad \boxed{V_{Th} = \frac{E R_2}{R_1 + R_2}}$$

To find R_{Th}:

The load resistance R_L is removed. The cell is disconnected and the wires are short as shown.

The effective resistance across $AB =$ Thevenin's resistance R_{Th}.



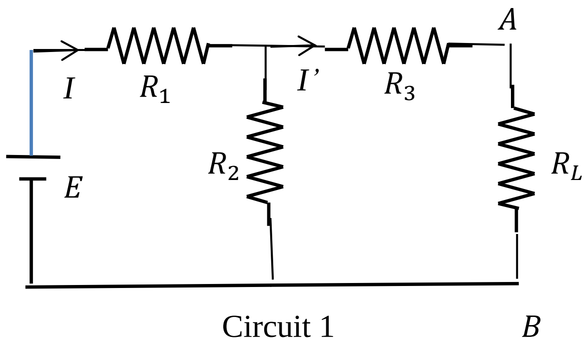
$$\boxed{R_{Th} = R_3 + \frac{R_1 R_2}{R_1 + R_2}}$$ [R_1 is parallel to R_2 and this combination in series with R_3]

If the cell has internal resistance r, then $V_{Th} = \frac{E R_2}{R_1 + R_2 + r}$ and $R_{Th} = R_3 + \frac{(R_1 + r) R_2}{R_1 + r + R_2}$.

<u>Proof of Thevenin's theorem:</u>

Consider the network as shown in circuit 1.

The equivalent thevenin's circuit is as shown in circuit 2.



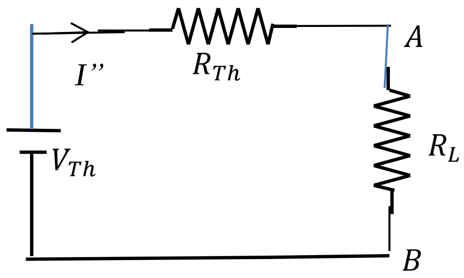
Circuit 1

The effective resistance of the network in (1) is R_3 and R_L in series and this combination is parallel to R_2 which in turn is in series with R_1.

Thus, $R_{eff} = R_1 + \frac{R_2 (R_3 + R_L)}{R_2 + R_3 + R_L}$ -------------- (1)

The current I in the circuit is

$$I = \frac{E}{R_{eff}} = \frac{E}{R_1 + \frac{R_2 (R_3 + R_L)}{R_2 + R_3 + R_L}}$$



Circuit 2

or $I = \frac{E(R_2 + R_3 + R_L)}{R_1 R_2 + R_1 R_3 + R_1 R_L + R_2 R_3 + R_2 R_L}$ ----------- (2)

The current through the load resistance (I') is found using branch current method.

$I' = \frac{I R_2}{R_2 + R_3 + R_L}$ ----------- (3)

Substituting for I from (2) in (3)

$$I' = \frac{E\left(R_2 + R_3 + R_L\right)R_2}{\left(R_2 + R_3 + R_L\right)\left(R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L\right)}$$

or $I' = \dfrac{ER_2}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L}$ ------------ (4)

Thevenin's voltage $V_{Th} = \dfrac{ER_2}{R_1 + R_2}$ ----------- (5)

Thevenin's resistance $R_{Th} = R_3 + \dfrac{R_1R_2}{R_1 + R_2}$ ---------- (6)

Consider the equivalent circuit (circuit (2))

The current I'' in the equivalent circuit is $I'' = \dfrac{V_{Th}}{R_{Th} + R_L}$ ----------- (7)

Substituting for V_{Th} and R_{Th} from (5) and (6) in (7)

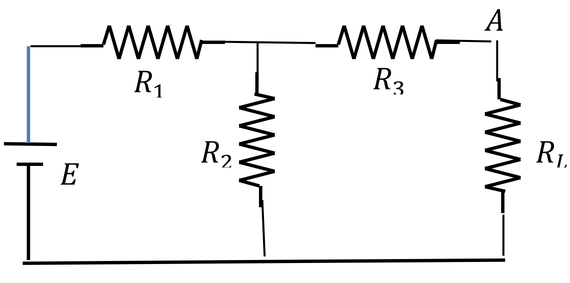
$$I'' = \frac{ER_2}{R_1 + R_2} \times \frac{1}{R_3 + \dfrac{R_1R_2}{R_1 + R_2} + R_L} = \frac{ER_2}{\left(R_1 + R_2\right)\dfrac{R_3R_1 + R_3R_2 + R_1R_L + R_2R_L + R_1R_2}{\left(R_1 + R_2\right)}}$$

or $I'' = \dfrac{ER_2}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L}$ ------------- (8)

From equations (4) and (8), it is observed that $I' = I''$.
Hence Thevenin's theorem is verified.

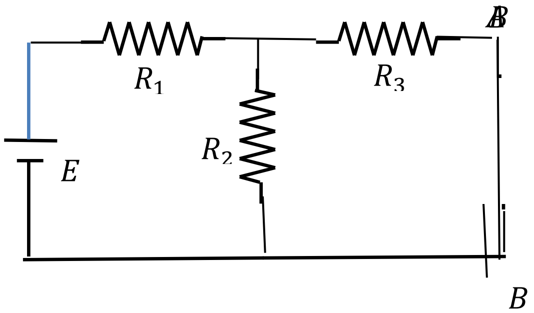
2 Norton's theorem

Statement: Any linear network can be replaced by an equivalent network having a single constant current source called Norton's current I_N and a single parallel resistance called Norton's resistance R_N.



Explanation: Consider the network as shown in the circuit diagram. Let E be the emf of the source.

To find I_N : The load resistance is disconnected from the terminals A and B. The two terminals are short as shown.



The effective resistance as seen by the terminals A and B is given by

$R_{eff} = R_1 + \frac{R_2\, R_3}{R_2 + R_3}$ or $R_{eff} = \frac{R_1 R_2 + R_1 R_3 + R_2\, R_3}{R_2 + R_3}$

The current driven by the source is

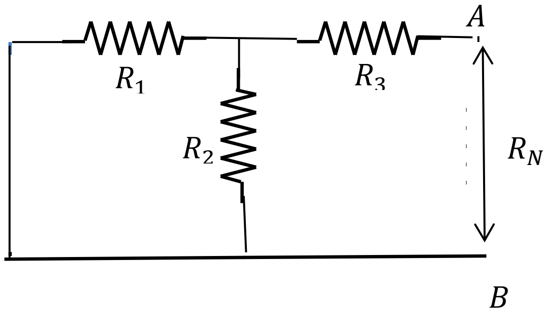
$$I = \frac{E}{R_{eff}} = \frac{E}{\frac{R_1 R_2 + R_1 R_3 + R_2\, R_3}{R_2 + R_3}} = \frac{E(R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_2\, R_3}$$

The short circuit current is the Norton current = I_N = current through R_3

$$I_N = I \times \frac{R_2}{R_2 + R_3} = \frac{E(R_2 + R_3)}{R_1 R_2 + R_1 R_3 + R_2\, R_3} \times \frac{R_2}{R_2 + R_3}$$

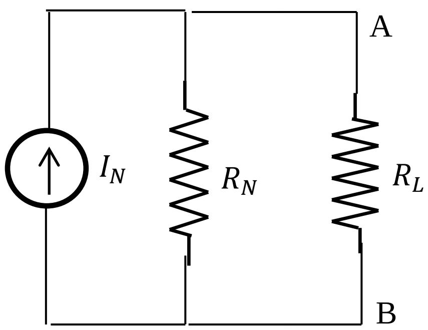
Thus $I_N = \frac{E\, R_2}{R_1 R_2 + R_1 R_3 + R_2\, R_3}$

To find R_N :



The load resistance is removed across AB and the circuit is open at these terminals. The battery is removed and is replaced by a short circuit as shown. The internal resistance of the battery is assumed to be zero. The Norton resistance is given by the effective resistance at the terminals AB. It is given by $R_N = R_3 + \frac{R_1 R_2}{R_1 + R_2}$.

By finding I_N and R_N the Norton equivalent circuit is drawn as shown.

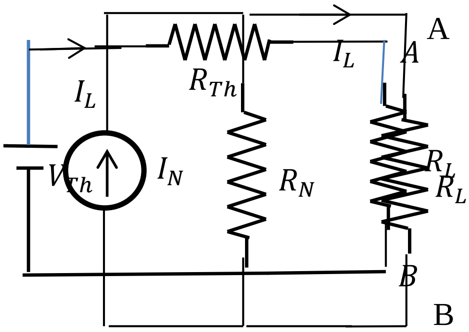


Duality of Thevenin's and Norton's equaivalent circuits

Consider the Thevenin's and Norton's equivalent circuits as shown in the circuit diagrams below.The thevenin's circuit consists of a source V_{Th} and a series resistance R_{Th} . According to Norton's circuit this circuit is equal to a single current source given by $I_N = \frac{V_{Th}}{R_{Th}}$ and the resistance connected in parallel i.e. $R_N = R_{Th}$.

The current flowing through the load resistance in the thevenin's equivalent circuit is given by $I_L = \frac{V_{Th}}{R_{Th} + R_L}$ ……(1)

The current flowing through the load resistance in the Norton's equivalent circuit is given by $I_L = \frac{I_N\, R_N}{R_N + R_L}$ .....(2)



As $I_N = \frac{V_{Th}}{R_{Th}}$, equation (2) becomes $I_L = \frac{\frac{V_{Th}}{R_{Th}} R_N}{R_N + R_L}$

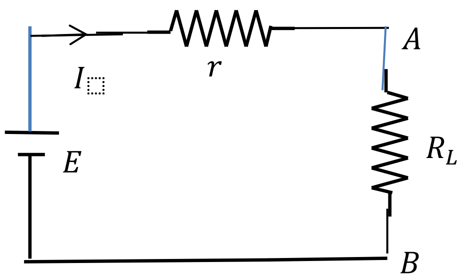
Also $R_N = R_{Th}$. the above equation is $I_L = \frac{\frac{V_{Th}}{R_{Th}} R_{Th}}{R_{Th} + R_L}$ or $I_L = \frac{V_{Th}}{R_{Th} + R_L}$ .....(3)

Comparing equations (1) and (3) it is clear that the two equations are the same. i.e. the current flowing through the load resistances in the two circuits are the same. Thus the two circuits are equivalent.

3. Maximum power Transfer Theorem

Statement: The power transferred by a source to the load resistance in a network is maximum when the load resistance is equal to the internal resistance of the source.

Proof of Maximum power transfer theorem:



Consider a network with a source of emf E and internal resistance r connected to a load resistance R_L. The current I in the circuit is

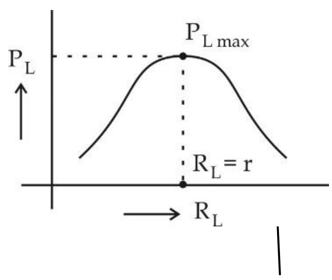
$$I = \frac{E}{R_L + r} \text{ ----------- (1)}$$

The power delivered to load resistance R_L is

$$P_L = I^2 R_L \text{ or } P_L = \left(\frac{E}{R_L + r}\right)^2 R_L$$

(from equation (1))

$$P_L = \frac{E^2}{(R_L + r)^2} R_L \text{ ---------- (2)}$$



The variation of P_L with R_L is as shown.

P_L is found to be maximum for a particular value of R_L when P_L is maximum, $\frac{dP_L}{dR_L}=0$

[$\because$ No variation of P_L with R_L at $P_{L\max}$]

i.e., $\frac{d}{dR_L}\left(\frac{E^2R_L}{(R_L+r)^2}\right)=0$ or $\frac{d}{dR_L}\left[E^2R_L(R_L+r)^{-2}\right]=0$

Differentiating $E^2\left[R_L(-2)(R_L+r)^{-3}+(R_L+r)^{-2}\right]=0$ or $\frac{-2R_L}{(R_L+r)^3}+\frac{1}{(R_L+r)^2}=0$ or

$\frac{2R_L}{(R_L+r)^3}=\frac{1}{(R_L+r)^2}$

Thus, $\frac{2R_L}{R_L+r}=1 \quad \Rightarrow \quad 2R_L=R_L+r$ or $\boxed{R_L=r}$

Thus the power delivered to the load resistance is maximum when the load resistance is equal to the internal resistance of the source.

To show that the maximum power transfer efficiency of a circuit is 50%

The power across the load $P_L=I^2R_L=\frac{E^2}{(R_L+r)^2}R_L$ ----------- (1)

From the maximum power transfer theorem, P_L is maximum when $R_L=r$. Putting this condition in equation (1),

$P_{L\max}=\frac{E^2}{(2R_L)^2}R_L \Rightarrow P_{L\max}=\frac{E^2}{4R_L}$ ------------- (2)

The power that is taken from the voltage source is (or power generated by the source),

$P=I^2(R_L+r)=\frac{E^2}{(R_L+r)^2}(R_L+r)$ or $P=\frac{E^2}{R_L+r}$. When $R_L=r$,

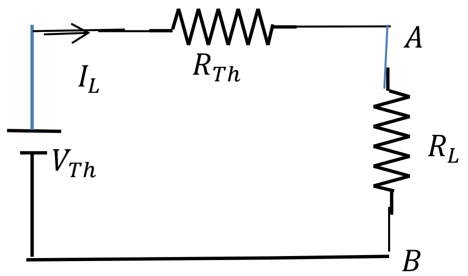
$P=\frac{E^2}{2R_L}$ -------- (3)

Dividing equation (2) by (3)

$\frac{P_{L\max}}{P}=\frac{E^2}{4R_L}\times\frac{2R_L}{E^2}=\frac{1}{2}$ or $P_{L\max}=\frac{P}{2}$

Thus the maximum power delivered to the load is only half the power generated by the source or the maximum power transfer efficiency is 50%. The remaining 50% power is lost across the internal resistance of the source.

To show that the emf of the source is $E = \sqrt{4\,P_{max}R_{Th}}$



Consider the thevenin's equivalent circuit as shown. The current flowing through the load resistance is given by $I_L = \frac{V_{Th}}{R_{Th}+R_L}$ ....(1)

The power delivered to the load is given by

$P_L = I_L^2\, R_L$ ....(2)

Substituting for I_L from (1)in (2), we get $P_L = \left(\frac{V_{Th}}{R_{Th}+R_L}\right)^2 R_L$ ....(3)

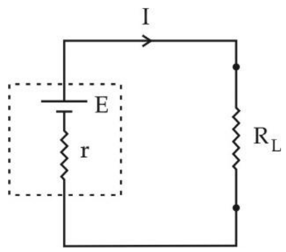
The power delivered to the load is maximum when the load resistance is equal to the internal resistance i.e. thevenin's resistance $R_L = R_{Th}$

Thus maximum power is given by (putting the above condition in (3))

$P_{max} = \left(\frac{V_{Th}}{R_{Th}+R_{Th}}\right)^2 R_{Th}$ or $P_{max} = \left(\frac{V_{Th}}{2R_{Th}}\right)^2 R_{Th}$

$P_{max} = \frac{{V_{Th}}^2}{4R_{Th}}$ or ${V_{Th}}^2 = 4\,P_{max}R_{Th}$ or $\boldsymbol{V_{Th} = \sqrt{4\,P_{max}R_{Th}}}$

If the circuit is represented by source of emf E, then $\boldsymbol{E = \sqrt{4\,P_{max}R_{Th}}}$



Note : Under the condition of maximum power transfer, the load voltage is one half of the open circuit voltage at the load terminals, i.e. source voltage.

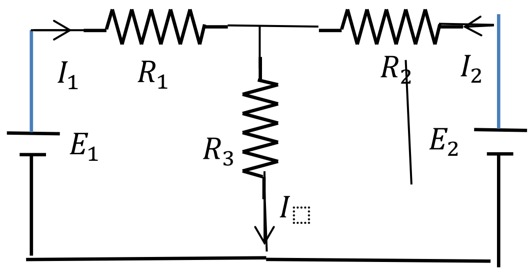
$V_L = I\,R_L$ as $I = \frac{E}{R_L+r}$, thus $V_L = \frac{E}{R_L+r} R_L$

As $R_l = r$ when the power delivered is maximum, thus $V_L = \frac{E}{2R_L} R_L = \frac{E}{2}$.

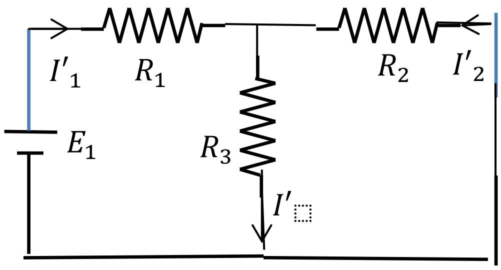
4. Superposition theorem

Statement: In a linear network having number of voltage or current sources and resistances, the current through any branch of the network is the algebraic sum of the currents due to each of the sources when acting independently.

Explanation: By mesh current analysis.

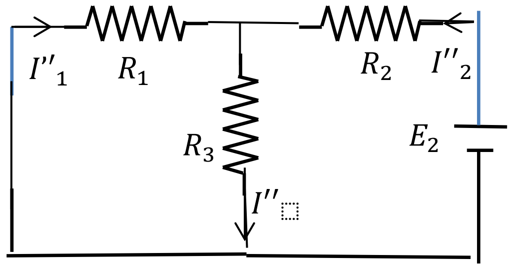


1. Consider the network as shown. The currents in different branches of the network are I_1, I_2 and I as shown. Also $I_1 + I_2 = I$.



2 Let the internal resistance r of the cells be negligible].

The cell E_2 is removed and the terminals are short as shown. Now the currents in the branches are I_1', I_2' and I'. Also $I' = I_1' + I_2'$.



3 The E_1 is removed and the terminals are short as shown. The currents are I_1'', I_2'' and I''. Also $I'' = I_1'' + I_2''$.

According to superposition theorem $I_1 = I_1' + I_1''$. $I_2 = I_2' + I_2''$ and $I = I' + I''$

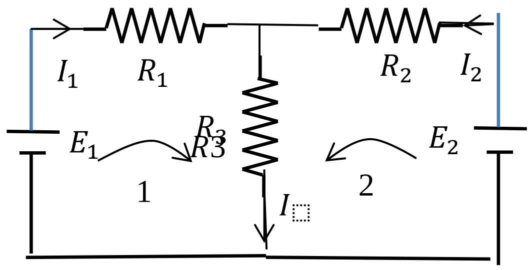
$\boxed{I = I_1 + I_2}$

Verification of superposition theorem:

1. Consider the network shown. Applying Kirchhoff's voltage to the loop 1.

$I_1R_1 + I_1R_3 + I_2R_3 = E_1 \qquad [\because I_1R_1 + I(R_3) = E_1 \qquad I = I_1 + I_2]$

or $I_1 = \dfrac{E_1 - I_2R_3}{R_1 + R_3}$ ----------- (1)



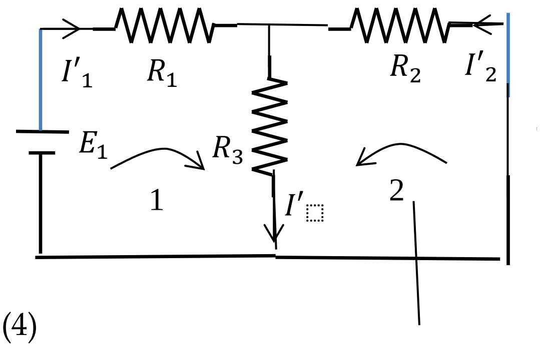
Considering loop 2, $I_2R_2 + IR_3 = E_2$.

$I_2R_2 + I_1R_3 + I_2R_3 = E_2$

$I_2 = \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ---------- (2)

Thus, $I = I_1 + I_2$

$I = \dfrac{E_1 - I_2R_3}{R_1 + R_3} + \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ----------- (3)



2. Consider the circuit shown with E_2 removed and terminals short. Applying Kirchhoff's law to loop 1.

$I_1'R_1 + I'R_3 = E_1 \qquad$ As $\quad I' = I_1' + I_2'$,

$I_1R_1 + I_1'R_3 + I_2'R_3 = E_1 \Rightarrow \quad I_1' = \dfrac{E_1 - I_2'R_3}{R_1 + R_3}$ ---------- (4)

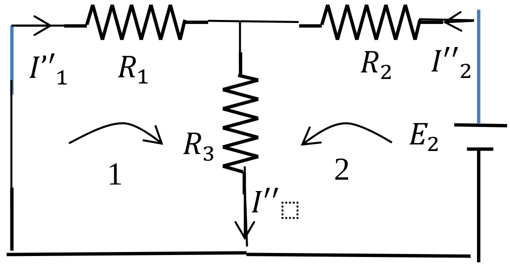
Similarly for loop 2,

$I_2'R_2 + I'R_3 = 0 \quad \Rightarrow \quad I_2'R_2 + I_1'R_3 + I_2'R_3 = 0$

$$I_2' = -\frac{I_1'R_3}{R_2 + R_3} \text{ ----------- (5)}$$

$$I' = I_1' + I_2' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3} \text{ ....(6)}$$

3. Consider the circuit with E_1 removed and terminals short.



For loop (1) $I_1''R_1 + I''R_3 = 0$

As $I'' = I_1'' + I_2''$

$$I_1''R_1 + I_1''R_3 + I_2''R_3 = 0 \quad \Rightarrow \quad I_1'' = \frac{-I_2''R_3}{R_1 + R_3} \text{ ------------ (7)}$$

For loop (2)

$I_2''R_2 + I''R_3 = E_2 \quad \Rightarrow \quad I_2''R_2 + I_1''R_3 + I_2''R_3 = E_2$

or $I_2'' = \dfrac{E_2 - I_1''R_3}{R_2 + R_3}$ ------------- (8)

$$I'' = I_1'' + I_2'' = \frac{-I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3} \text{ -------------- (9)}$$

Adding equations (6) and (9)

$$I' + I'' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3} - \frac{I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3}$$

$$= \frac{1}{R_1 + R_3}\left[E_1 - I_2'R_3 - I_2''R_3\right] + \frac{1}{R_2 + R_3}\left[E_2 - I_1''R_3 - I_1'R_3\right]$$

$$I' + I'' = \frac{1}{R_1 + R_3}\left[E_1 - R_3\left(I_2' + I_2''\right)\right] + \frac{1}{R_2 + R_3}\left[E_2 - R_3\left(I_1' + I_1''\right)\right] \text{ ....(10)}$$

Comparing equations (3) and (10) it is observed that

$I_1 = I_1' + I_1''$

$I_2 = I_2' + I_2''$

$I = I' + I''$

Hence the proof of the theorem.

5. Reciprocity theorem

Reciprocity theorem is one of the DC network analysis or AC network analysis technique and deals with the relationship between a voltage source in a part of the circuit and it's response at some other part of the circuit.

Statement : In any linear network, If a source of emf (E) acting in a branch (let "A") of the circuit produces the current "I" in another branch (let "B") of the circuit. Then when the emf (E) acts in the second branch ("B") , it will produce the same current "I" in the first branch ("A").

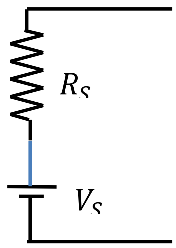
In another words, the supply voltage "E" and current "I" are mutually transferable in any linear circuit. The ratio between the Voltage and the Current that are mutually transferable is called the transfer resistance.

PART A

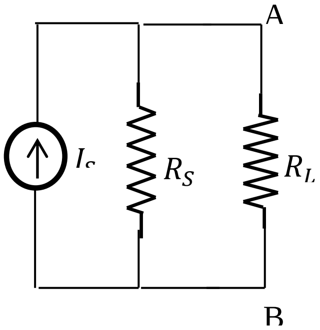
1 (a) Explain the concept of voltage source and current sources.
   (b) Establish the equivalence (duality) between the voltage and current sources.
2 (a) Distinguish between an ideal voltage source and an ideal current source.
   (b) Explain the process of converting an voltage source in to a current source and a current source to voltage source?
3 State and prove Thevenin's theorem.
4 State and prove Superposition theorem.
5 (a) State and prove Maximum power transfer theorem
   (b) Show that the maximum power transferred to the load is half the power generated by the source.
6 (a) State and explain Norton's theorem.
   (b) Explain the duality of Norton's and Thevenin's equivalent circuits.
7 (a) State and explain maximum power transfer theorem.
   (b) Show that emf of the source is $E = \sqrt{4\,P_{max}R_{Th}}$
   (c) Show the the voltage at the loadis half the supply voltage when the power delivered is maximum.
8 (a) State and explain Kirchhoff's current law and voltage law.
   (b) Explain Reciprocity theorem.

PART B

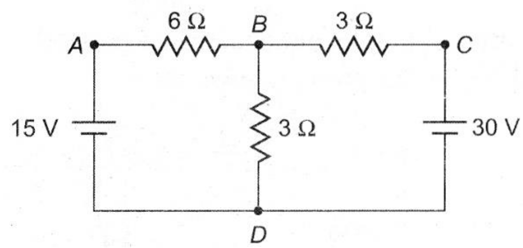
1 A voltage source representation is as shown. Find the equivalent current source circuit. Given $V_S = 12V$ and $R_S = 4k\Omega$.



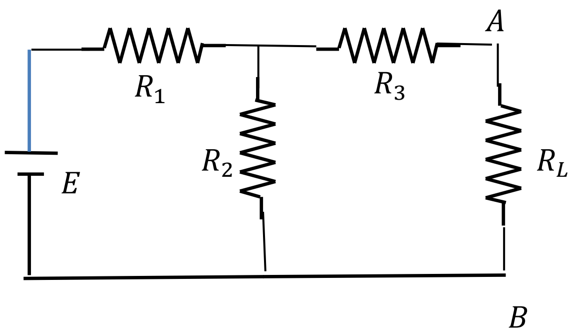
2 Find the current in the $5k\Omega$ resistor by converting the current source to the voltage source in the given circuit. Given $I_S = 10mA$, $R_S = 2k\Omega$ and $R_L = 5k\Omega$

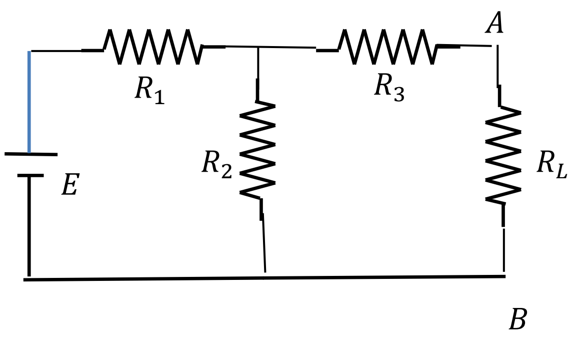


3 In the circuit shown in figure, find the current through the branch BD using Kirchhoff's laws



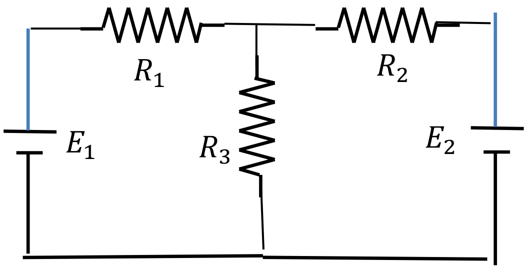
4 In the network shown, find the current through R_L using Thevenin's theorem Given $R_1 = 6\Omega$, $R_2 = 3\Omega$, $R_3 = 4\Omega$, $R_L = 5\Omega$ and E = 12V



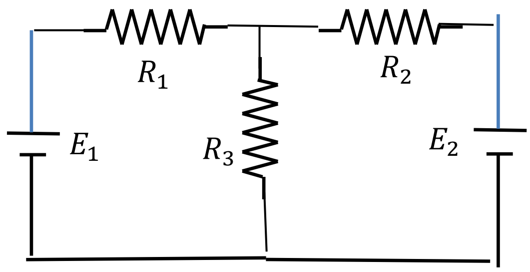


5 Using Thevenin's theorem, find the pd across AB for the circuit Given $R_1 = 5\Omega$, $R_2 = 2\Omega$, $R_3 = 4\Omega$, $R_L = 20\Omega$ and E = 50V.

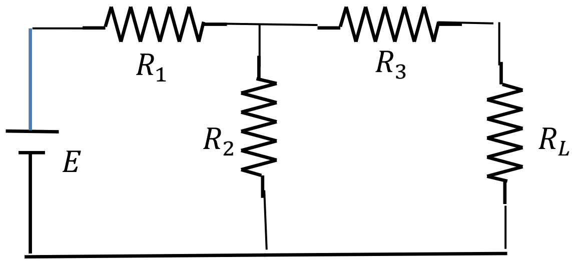
6 Find the current through 8Ω resistor using the superposition theorem in the given circuit $R_1 = 4\Omega$, $R_2 = 6\Omega$, $R_3 = 8\Omega$, $E_1 =$ $6V$ and E_2 = 8V.



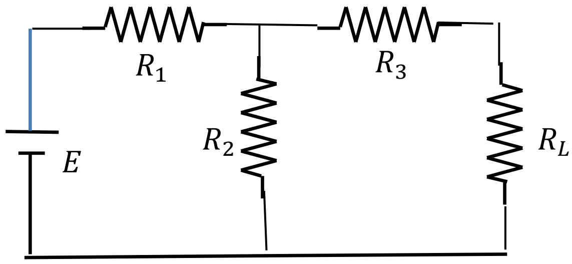
Find the current in different resistances of the circuit using the superposition theorem $R_1 = 20\Omega$, $R_2 = 30\Omega$, $R_3 = 20\Omega$, $E_1 = 32V$ and E_2 = 20V.



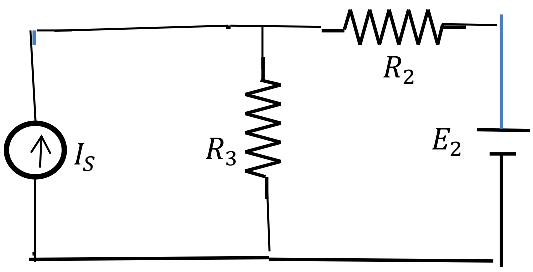
8 Find the thevenin's equivalent circuit for the network shown below and find the current through $R_L = 6\Omega$.
$R_1 = 3\Omega$, $R_2 = 6\Omega$, $R_3 = 2\Omega$, $R_L = 6\Omega$ and E = 18V



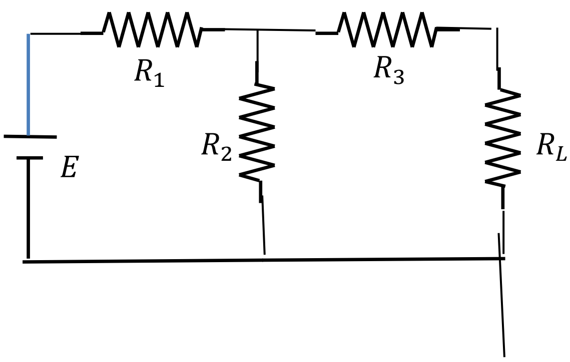
9 For the network shown below, determine the value of R_{Th} for maximum power to R_L and calculate the power delivered under these conditions
$R_1 = 6\Omega$, $R_2 = 3\Omega$, $R_3 = 8\Omega$, and E = 12V



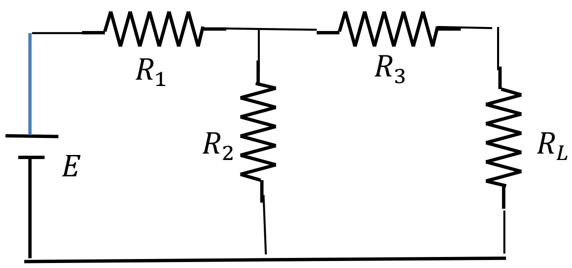
10 Using superposition theorem find the current through $1k\Omega$ in the circuit shown.
$R_2 = 1k\Omega$, $R_3 = 3k\Omega$, $I_S = 8mA$ and E_2 = 32V



11 Calculate the current through 10Ω resistor using Norton's theorem for the circuit shown.
$R_1 = 2\Omega$, $R_2 = 5\Omega$, $R_3 = 3\Omega$, and E = 30V

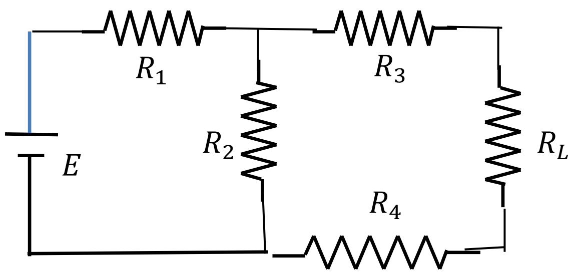


12 Draw the Thevenin's and Norton's equivalent circuits for the network shown below and fid the current through the load in each case. $R_1 = 6\Omega$, $R_2 = 8\Omega$, $R_3 = 8\Omega$, and E = 24V $R_L = 32\Omega$.

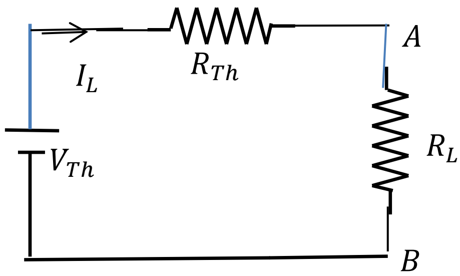


13 Find the value of R_L required to obtain maximum powe in the circuit shown. Also fine the value of maximum power.

$R_1 = 300\Omega$, $R_2 = 100\Omega$, $R_3 = 25\Omega$, and E = 100V $R_4 = 50\Omega$.



PART C



1 Is current in the circuit shown below maximum or minimum when the load resistance is short circuited?

2 Under what condition a real voltage source be imagined to be an ideal voltage sourec?

3 Under what condition a real current source be imagined as an ideal current source?

4 Why the internal resistance of an ideal voltage source is zero? Explain.

5 What is the maximum power delivered across the load with respect to the power generated at the source?

Transient currents

Syllabus : Transient currents : Self inductance – definition, explanation, expression $L = \frac{\mu N^2 A}{l}$; Magnetic field energy stored in an inductor; Growth and decay of charge in series RC circuit, Growth and decay of current in series LR circuit, Decay of charge in series LCR circuit - Damped, under-damped and over-damped conditions

Self Inductance: The phenomenon in which emf is induced in a coil of a circuit when the magnetic field linked with it changes is called **electromagnetic induction**.

Faraday's laws of electromagnetic induction:

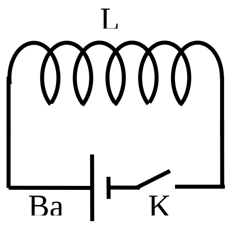
I Law: Whenever there is a change in the magnetic flux linked with a coil, emf and hence current is induced in the circuit. The induced emf will last till the magnetic flux linked with the circuit keeps changing.

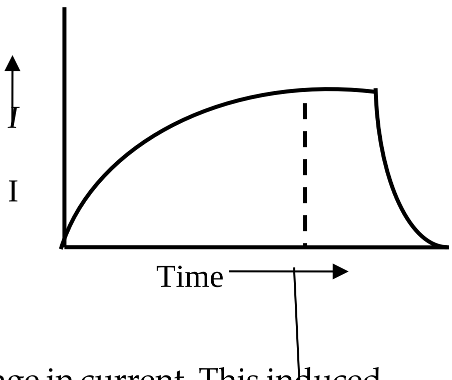
II Law: The magnitude of induced emf (e) is directly proportional to the rate of change of magnetic flux (φ) linked with the circuit. $e \propto \frac{d\varphi}{dt}$

Lenz's law: Whenever an induced emf is set up, the direction of the induced emf is such as to oppose the cause producing it.

Introducing the feature of Lenz's law in the expression of Faraday's II law, we get $e \propto -\frac{d\varphi}{dt}$ or $e = -K\frac{d\varphi}{dt}$. It can be shown that in SI units K = 1. Thus $e = -\frac{d\varphi}{dt}$.

Self induction:





Consider a coil connected in series with a battery and a tap key K. When the key is closed, the current in the circuit grows from zero to maximum in a finite time as shown in the graph. This growth of current produces varying magnetic flux linked with coil giving rise to induced emf in the coil.

According to Lenz's law, the induced emf opposes the change in current. This induced emf is called back emf. Emf is also induced when the circuit breaks.

The phenomenon of emf being induced in a coil due to change of current in the same coil is called **self induction.**

The magnetic flux linked with coil is directly proportional to the current in the coil.

Thus $\varphi \propto I$ or $\varphi = LI$ where L is a constant called the **self inductance** of the coil and depends on the 1) geometry, 2) number of turns and 3) material of the core of the coil. Also L can be shown to be equal to $L = \mu_r \mu_0 N^2 A/l$.

The induced emf is $e = -\frac{d\varphi}{dt}$ or $e = -\frac{d(LI)}{dt}$ *or* $e = -L\frac{dI}{dt}$.

If $\frac{dI}{dt} = 1$ then **e = - L.** Thus self inductance of a coil is numerically equal to the induced emf in a coil when current in the coil changes at the rate of 1 ampere per second.

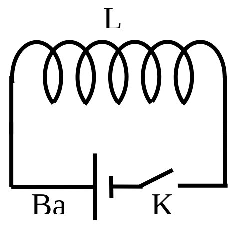
The unit of self inductance is **henry (H).**

In the equation $e = -L\frac{dI}{dt}$, if $\frac{dI}{dt} = 1\,\text{As}^{-1}$ and e = 1volt, then L = 1H.

Thus the self inductance of a coil is said to be **1 henry**, if 1 volt of emf is induced in the coil when the current in it changes at the rate of 1ampere per second.

To arrive at the expression for self inductance $L = \frac{\mu N^2 A}{l}$

Consider an inductance coil in the form of a solenoid as shown. The magnetic field along the axis of this solenoid is given by $B = \mu_0 nI = \frac{\mu_0 NI}{l}$ ….(1)



where n is the number of turns per unit length given by $n = \frac{N}{l}$

where l is the length of the solenoid.

The magnetic flux through each turn of the solenoid is given by $\phi = B\,A$ ….(2)

where A is the area of cross section of the solenoid. Thus substituting for B from(1) in (2) $\phi = \left(\frac{\mu_0 NI}{l}\right) A$

Total magnetic flux linked with all the N turns of the solenoid is given by $N\phi$.

Thus $N\phi = \frac{\mu_0 N^2 A I}{l}$ …..(3)

From the principle of electromagnetic induction the induced emf in the coil due to change in current is $e = -L\frac{dI}{dt}$ Also e = -- $\frac{d}{dt} N\phi = -\frac{\mu_0 N^2 A}{l}\frac{dI}{dt}$

Comparing the above equations we have $L\frac{dI}{dt} = \frac{\mu_0 N^2 A}{l}\frac{dI}{dt}$

Simplifying, the self inductance of the solenoid is $L = \frac{\mu_0 N^2 A}{l}$

If the solenoid is wound on a material whose relative permeability is μ_r , then the expression for L is $L = \frac{\mu_0\, \mu_r N^2 A}{l}$ or $\boldsymbol{L = \frac{\mu\, N^2 A}{l}}$ where $\mu = \mu_0\, \mu_r$ called the absolute permeability of the medium.

Expression for energy stored in a coil:

Consider a coil of self inductance *L* carrying a current *I*. When this current is increased, the magnetic flux linked with coil changes giving rise to induced emf in the coil called back emf.

According to Lenz's law, the induced emf opposes the increase of current. Thus work has to be done in opposing the back emf. This work done is stored as potential energy in the magnetic field around the coil.

Let e be the induced emf developed in the coil at an instant of time t. Then the emf can be represented by the equation $e = -L\frac{dI}{dt}$ ….. (1)

If dW is the small amount of work done in an interval of time dt, then

$dW = e\, I\, dt$ ….(2) $[By\ definition\ rate\ of\ doing\ work\ done = product\ of\ voltage\ and\ current\ ]$

Substituting for e from (1) in (2), $dW = \left(-L\frac{dI}{dt}\right) I\, dt = LI\, dI$

Total work done in increasing the current from zero a maximum value I_m, the above equation is integrated as $\int dW = \int_0^{I_m} LI\, dI$

or $W = \frac{1}{2} LI_m^2$.

As the work done represents the energy stored in the coil in the form of magnetic field, we can write the equation as $U = \frac{1}{2} LI_m^2$. Here U represents the potential energy stored in the coil.

Note:

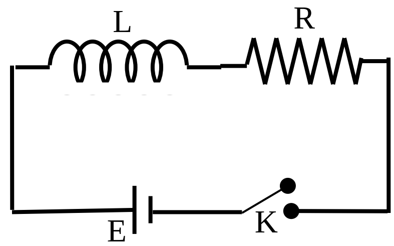
1. The physical significance of self inductance – It is called the electrical inertia similar to mass which is referred to as measure of inertia in a mechanical system. If mass is large then more is the opposition to motion, similarly greater is the inductance, larger is the back emf which opposes the growth of current.
2. Inductor is a passive element like resistor and capacitor in a circuit. The inductance L measures the opposition to change in current in a circuit. A n inductor allows direct current to pass through it.
3. Similar to the energy stored in a inductance coil as magnetic fields, energy is stored in a capacitor as electric fields. In case of capacitor, $U = \frac{1}{2} CV^2 = \frac{q^2}{2C}$.

Transient Currents :

The transient phenomena is one which exists for very short duration of time in which currents, changes and power vary for a short interval of time. In a circuit with only resistance, the current, voltage and power are constants. But in circuits with inductors and capacitors, the current, voltage, charge and power vary.

$L - R$ Circuit

1. Growth of current in a LR Circuit :



Consider a circuit containing an inductor of inductance L, resistor of resistance R and a steady dc source of emf E connected in series as shown.

When the key K is closed the current rises in the circuit which in turn generates the back emf e in the inductor given by $e = -L\frac{dI}{dt}$. This emf opposes the growth of current. After a short interval of time current reaches steady value. The voltage across the resistance is given by $V_R = IR$.

From Kirchhoff voltage law, the net emf is equal to voltage across R.

i.e. $E - L\frac{dI}{dt} = V_R$ or $E = IR + L\frac{dI}{dt}$ .....(1)

When the current reaches its maximum value $I = I_0$, the rate of change of current is zero, $\frac{dI}{dt} = 0$. Thus back emf becomes zero, i,e, $L\frac{dI}{dt} = 0$.

Equation (1) now is = I_0R . Thus equation (1) becomes $I_0R = IR + L\frac{dI}{dt}$

Rearranging $R(I_0 - I) = L\frac{dI}{dt}$ or $\frac{dI}{(I_0-I)} = \frac{R}{L}\,dt$ …..(2)

Integrating (2), $\int\frac{dI}{(I_0-I)} = \frac{R}{L}\int dt$ or $-\,log(I_0 - I) = \frac{R}{L}\,t + K$ …..(3)

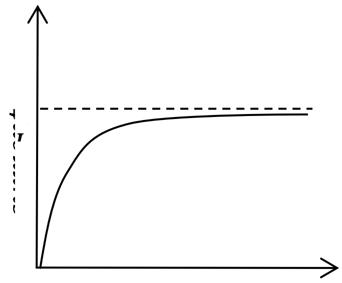
where K is the constant of integration. By using initial conditions K can be evaluated. When $t = 0, I = 0$. The equation (3) becomes $-\,logI_0 = K$

Equation (3) now is $-\,log(I_0 - I) = \frac{R}{L}\,t - logI_0$ or $log(I_0 - I) = -\frac{R}{L}\,t + logI_0$

Thus $log(I_0 - I) - logI_0 = -\frac{R}{L}\,t$ or $log\frac{(I_0-I)}{I_0} = -\frac{R}{L}\,t$ or $\frac{(I_0-I)}{I_0} = e^{-\frac{R}{L}t}$

$1 - \frac{I}{I_0} = e^{-\frac{R}{L}t}$ or $\frac{I}{I_0} = 1 - e^{-\frac{R}{L}t}$ or $\boldsymbol{I = I_0\left(1 - e^{-\frac{R}{L}t}\right)}$ **….(4)**

Equation (4) shows the growth of current with time in a L – R circuit. Thus when $t = 0$, $I = 0$ and when $t = \infty$, $I = I_0$. The variation is represented graphically as shown which is exponential.

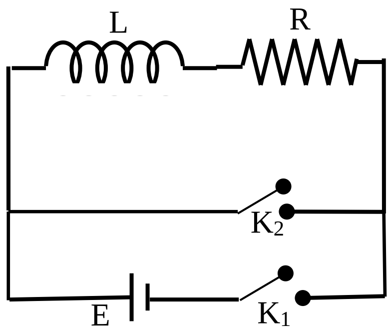


. When = $\boldsymbol{\tau} = \frac{\boldsymbol{L}}{\boldsymbol{R}}$, equation (4) becomes

$I = I_0(1 - e^{-1}) = I_0\left(1 - \frac{1}{e}\right) = I_0\left(1 - \frac{1}{2.718}\right) = 0.632\, I_0$

. Thus τ is the time constant defined as the time required by the current to reach 63.2% of its maximum current. It is observed that when = $5\frac{L}{R}$, the current reaches its steady state of I_0 .

2. Decay of current in a LR Circuit :



Consider a circuit containing an inductor of inductance L, resistor of resistance R and a steady dc source of emf E connected in series as shown.

When the key K_1 is closed and K_2 open, the current grows in the circuit and reaches maximum value. Now key K_1 is opened (E = 0) and key K_2 is closed. The current decreases in the circuit which in turn generates the back emf (e) in the inductor given by $e = -L\frac{dI}{dt}$. This emf opposes the decay of current. After a short interval of time current becomes zero.

From Kirchhoff voltage law, $E = IR + L\frac{dI}{dt}$

As E = 0, we have $0 = IR + L\frac{dI}{dt}$ ….(1) where the voltage across the resistor is $V_R = IR$

Rearranging the above equation, $\frac{dI}{I} = -\frac{R}{L}\, dt$ …..(2)

Integrating (2), $\int \frac{dI}{I} = -\frac{R}{L} \int dt$ or $\log I = -\frac{R}{L}\, t + K$ …..(3)

where K is the constant of integration. By using initial conditions K can be evaluated.

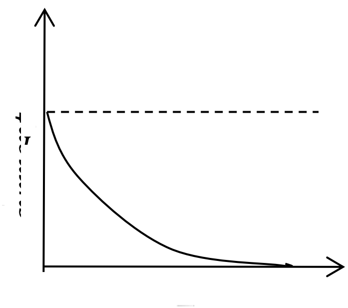
When $t = 0,\ I = I_0$. The equation (3) becomes $logI_0 = K$

Equation (3) now is $logI = -\frac{R}{L}\, t + logI_0$

Thus $logI - logI_0 = -\frac{R}{L}\, t$ or $log\frac{I}{I_0} = -\frac{R}{L}\, t$ or $\frac{I}{I_0} = e^{-\frac{R}{L}t}$

or $\boldsymbol{I = I_0 e^{-\frac{R}{L}t}}$ **….(4)**

Equation (4) shows the decay of current with time in a L – R circuit. Thus when $t = 0$, $I = I_0$ and when $t = \infty,\ I = 0$. The variation is represented graphically as shown which is exponential.

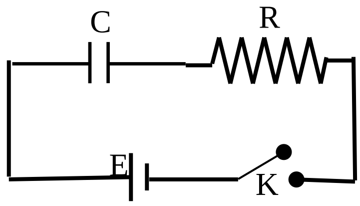


. When $= \boldsymbol{\tau} = \frac{\boldsymbol{L}}{\boldsymbol{R}}$, equation (4) becomes

$I = I_0 e^{-1} = I_0\left(\frac{1}{e}\right) = I_0\left(\frac{1}{2.718}\right) = 0.368\, I_0$. Thus τ is the time constant defined as the time required by the current to reach 36.8% of its maximum current. It is observed that when $= 5\frac{L}{R}$, the current becomes zero.

C − R Circuit

1. Growth of charge in a CR Circuit :



Consider a circuit containing a capacitor of capacitance C, resistor of resistance R and a steady dc source of emf E connected in series as shown.

When the key K is closed, the capacitor gets charged and the process continues till the capacitor gets completely charged. At this point the charging current becomes zero. At any instant the charge on the capacitor is $= CV_C$. Thus $V_C = \frac{q}{C}$. The voltage across the resistance is given by $V_R = IR$.

From Kirchhoff voltage law, $E = IR + \frac{q}{C}$ …..(1)

When the charge reaches its maximum value $q = q_0$, the current is zero, i.e. $IR = 0$.

$E = \frac{q_0}{C}$. Putting this condition in (1) $\frac{q_0}{C} = IR + \frac{q}{C}$ …..(2)

As $= \frac{dq}{dt}$, equation (2) is $\frac{q_0}{C} = R\frac{dq}{dt} + \frac{q}{C}$

Rearranging $\frac{q_0}{C} - \frac{q}{C} = R\frac{dq}{dt}$ or $\frac{q_0 - q}{C} = R\frac{dq}{dt}$ or $\frac{dq}{q_0 - q} = \frac{1}{CR}dt$ …..(3)

Integrating (3), $\int \frac{dq}{q_0 - q} = \frac{1}{CR}\int dt$ or $-\,log(q_0 - q) = \frac{1}{CR}\,t + K$ …..(4)

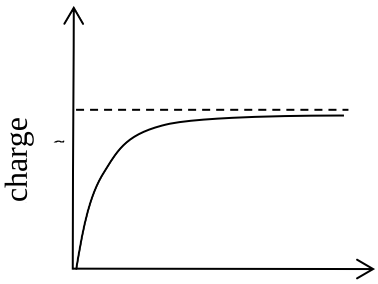
where K is the constant of integration. By using initial conditions K can be evaluated. When $t = 0,\ q = 0$. The equation (4) becomes $-\,log\,q_0 = K$

Equation (3) now is $-\,log(q_0 - q) = \frac{1}{CR}\,t - log\,q_0$ or $log(q_0 - q) = -\frac{1}{CR}\,t + log q_0$

Thus $log(q_0 - q) - log q_0 = -\frac{1}{CR}\,t$ or $log\frac{(q_0 - q)}{q_0} = -\frac{1}{CR}\,t$ or $\frac{(q_0 - q)}{q_0} = e^{-\frac{1}{CR}t}$

$1 - \frac{q}{q_0} = e^{-\frac{1}{CR}t}$ or $\frac{q}{q_0} = 1 - e^{-\frac{1}{CR}t}$ or $\boldsymbol{q = q_0\left(1 - e^{-\frac{1}{CR}t}\right)}$ **….(5)**

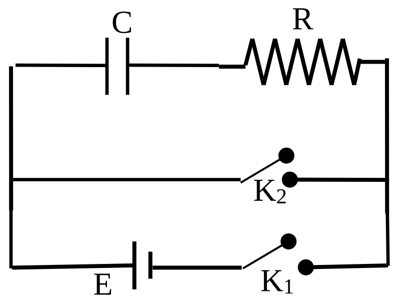
Equation (5) shows the growth of charge with time in a C – R circuit. Thus when $t = 0,\ q = 0$ and when $t = \infty,\ q = q_0$. The variation is represented graphically as shown which is exponential.



. When $= \boldsymbol{\tau} = CR$, equation (5) becomes

$q = q_0(1 - e^{-1}) = q_0\left(1 - \frac{1}{e}\right) = q_0\left(1 - \frac{1}{2.718}\right) =$ $0.632\,q_0$. Thus τ is the time constant defined as the time required by the charge to reach 63.2% of its maximum charge. It is observed that when $= 5CR$, the charge reaches its maximum value of q_0 . Dividing eqn. (5) by C on both sides, $\frac{q}{C} = \frac{q_0}{C}\left(1 - e^{-\frac{1}{CR}t}\right)$ or $V = V_0\left(1 - e^{-\frac{1}{CR}t}\right)$. Also $I = \frac{V}{R}$ and $I_0 = \frac{V_0}{R}$.

<u>2. Decay of charge in a CR Circuit :</u>



Consider a circuit containing a capacitor of capacitance C, resistor of resistance R and a steady dc source of emf E connected in series as shown.

When the key K_1 is closed and K_2 open, the charge grows in the circuit and reaches maximum value. Now key K_1 is opened (E = 0) and key K_2 is closed. The charge decreases in the circuit. After a short interval of time charge becomes zero.

From Kirchhoff voltage law, $E = IR + \frac{q}{C}$

As E = 0, we have $0 = IR + \frac{q}{C}$ ….(1) where the voltage across the resistor is $V_R = IR$

As $= \frac{dq}{dt}$, above equation is $0 = R\frac{dq}{dt} + \frac{q}{C}$ or $R\frac{dq}{dt} = -\frac{q}{C}$

Rearranging the above equation, $\frac{dq}{q} = -\frac{1}{CR}\,dt$ …..(2)

Integrating (2), $\int \frac{dq}{q} = -\frac{1}{CR}\int dt$ or $\log q = -\frac{1}{CR}t + K$ .....(3)

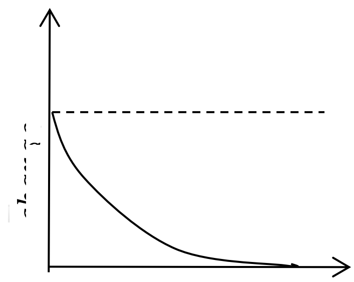
where K is the constant of integration. By using initial conditions K can be evaluated. When $t = 0,\ q = q_0$. The equation (3) becomes $log q_0 = K$

Equation (3) now is $log q = -\frac{1}{CR}t + log q_0$

Thus $log q - log q_0 = -\frac{1}{CR}t$ or $log\frac{q}{q_0} = -\frac{1}{CR}t$ or $\frac{q}{q_0} = e^{-\frac{1}{CR}t}$

or $\boldsymbol{q = q_0 e^{-\frac{1}{CR}t}}$ **....(4)** **or** $V = V_0\, e^{-\frac{1}{CR}t}$

Equation (4) shows the decay of charge with time in a C – R circuit. Thus when $t = 0$, $q = q_0$ and when $t = \infty,\ q = 0$. The variation is represented graphically as shown which is exponential.

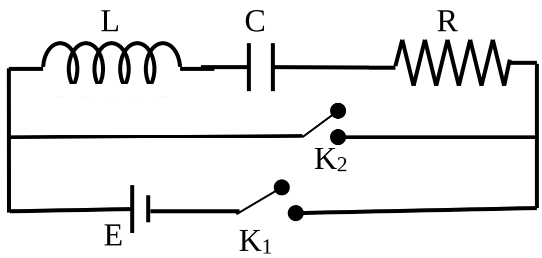


. When $= \boldsymbol{\tau = CR}$, equation (4) becomes

$q = q_0 e^{-1} = q_0\left(\frac{1}{e}\right) = q_0\left(\frac{1}{2.718}\right) = 0.368\, q_0$. Thus τ is the time constant defined as the time required by the charge to reach 36.8% of its maximum charge. It is observed that when t= $5CR$, the charge becomes zero.

LCR Series Circuit

Decay of charge in a LCR Series circuit



Consider a circuit containing an inductor of inductance L, a capacitor of capacitance C, resistor of resistance R and a steady dc source of emf E connected in series as shown.

When the key K_1 is closed, the capacitor gets charged to a value q_0. After some time, K_1 is opened (E = 0) and K_2 is closed. The charge gets discharged and becomes zero. Let q is the charge and I is the current at an instant of time t then if, V_L , V_C *and* V_R are the voltages across the inductor L, capacitor C and resistor R, from Kirchoff's law, we have

$0 = V_L + V_C + V_R = L\frac{dI}{dt} + \frac{q}{C} + IR$ or $L\frac{dI}{dt} + IR + \frac{q}{C} = 0$ ....(1)

As $I = \frac{dq}{dt}$ and $\frac{dI}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2}$, the above equation can be expressed as

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ Dividing throughout by , $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$

or $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$ ......(2) Let $\frac{R}{L} = 2b$ and $\frac{1}{LC} = k^2$.

If $q = x, then\ \frac{dq}{dt} = \frac{dx}{dt}$ and $\frac{d^2q}{dt^2} = \frac{d^2x}{dt^2}$

If the above conditions are put in equation (2), we get $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + k^2x = 0$ …..(3)

This is a second order differential equation. The possible solution of this equation is $x = e^{\alpha t}$, then $\frac{dx}{dt} = \alpha e^{\alpha t}$ and $\frac{d^2x}{dt^2} = \alpha^2 e^{\alpha t}$. Substituting these in equation (3), we get $\alpha^2 e^{\alpha t} + 2b\alpha e^{\alpha t} + k^2 e^{\alpha t} = 0$ or $\alpha^2 + 2b\alpha + k^2 = 0$

This is a quadratic equation in α with roots given by $\alpha = \frac{-2b\pm\sqrt{4b^2-4k^2}}{2} = -b \pm \sqrt{b^2 - k^2}$

As there are two roots $\alpha_1\ and\ \alpha_2$, the general solution is $x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$ ….(4)

where $\alpha_1 = -b + \sqrt{b^2 - k^2}$ and $\alpha_2 = -b - \sqrt{b^2 - k^2}$

Hence $x = q$ Thus $q = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$ …….(5)

To find $A\ and\ B$, initial conditions are used,

At $= 0$, $q = q_0$. Using this in (5) $q_0 = A + B$ or $A + B = q_0$ ….(6)

Differentiating (5) $\frac{dq}{dt} = A\alpha_1 e^{\alpha_1 t} + B\alpha_2 e^{\alpha_2 t}$

At $t = 0$, $\frac{dq}{dt} = 0$, The above equation is $0 = A\alpha_1 + B\alpha_2$

Putting $\alpha_1\ and\ \alpha_2$ in the above equation $A\left(-b + \sqrt{b^2 - k^2}\right) + B\left(-b - \sqrt{b^2 - k^2}\right) = 0$

$-Ab + A\sqrt{b^2 - k^2} - Bb - B\sqrt{b^2 - k^2} = 0$ or $-b(A + B) + \sqrt{b^2 - k^2}(A - B) = 0$

Thus $A - B = \frac{b(A+B)}{\sqrt{b^2-k^2}}$ or $A - B = \frac{q_0 b}{\sqrt{b^2-k^2}}$ ….(7) where $(A + B) = q_0$ …(6)

Adding (6) and (7) $2A = q_0 + \frac{q_0 b}{\sqrt{b^2-k^2}}$ or $A = \frac{1}{2}q_0\left[1 + \frac{b}{\sqrt{b^2-k^2}}\right]$ …(8)

Subtracting (6) and (7) $B = \frac{1}{2}q_0\left[1 - \frac{b}{\sqrt{b^2-k^2}}\right]$ …..(9)

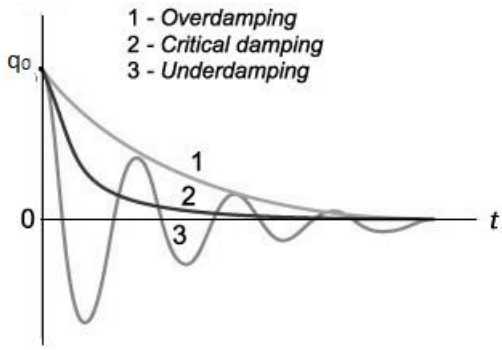
Substituting the values of A and B in (5), we get

$$q = \frac{1}{2}q_0\left[1 + \frac{b}{\sqrt{b^2-k^2}}\right]e^{(-b+\sqrt{b^2-k^2})t} + \frac{1}{2}q_0\left[1 - \frac{b}{\sqrt{b^2-k^2}}\right]e^{(-b-\sqrt{b^2-k^2})t}$$

$$\boldsymbol{q = \frac{1}{2}q_0 e^{-bt}\left\{\left(1 + \frac{b}{\sqrt{b^2-k^2}}\right)e^{\left(\sqrt{b^2-k^2}\right)t} + \left(1 - \frac{b}{\sqrt{b^2-k^2}}\right)e^{-\left(\sqrt{b^2-k^2}\right)t}\right\}} \quad \text{…..(10)}$$

This is the expression for decay of charge in a LCR series circuit.

Special cases:



Case 1. $\boldsymbol{b^2 < k^2}$, or $\boldsymbol{\frac{R^2}{4L^2} < \frac{1}{LC}}$ $\boldsymbol{or\ R < \sqrt{\frac{4L}{C}}}$ (**underdamped condition**) then $\sqrt{b^2 - k^2}$ is negative, i.e. imaginary.

We can express $\sqrt{b^2 - k^2}$ as $\sqrt{-(k^2 - b^2)} = i\omega \quad where \quad \omega = \sqrt{(k^2 - b^2)}$

Thus $\sqrt{b^2 - k^2} = i\omega$ .Now equation (10) can be expressed as

$$q = \frac{1}{2} q_0 e^{-bt} \left\{ \left(1 + \frac{b}{i\omega}\right) e^{i\omega t} + \left(1 - \frac{b}{i\omega}\right) e^{-i\omega t} \right\}$$

$$q = q_0 e^{-bt} \left\{ \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right) + \frac{b}{\omega}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2\,i}\right) \right\}$$ or $$q = q_0 \left[e^{-bt} \left(cos\omega t + \frac{b}{\omega} sin\omega t \right) \right]$$

$$q = q_0 \left[\frac{e^{-bt}}{\omega} (\omega cos\omega t + b sin\omega t) \right]$$ Let $b = k sin\theta$ and $\omega = k cos\theta$

Then $q = q_0 \left[\frac{e^{-bt}}{\omega} (k cos\omega t\, cos\theta + k sin\omega t\, sin\theta) \right]$ or $\boldsymbol{q = q_0 \left[\frac{k e^{-bt}}{\omega} cos(\omega t + \theta) \right]}$

The above equation indicates the decay of charge as damped oscillations. The charge oscillates from a maximum of q_0 till it becomes zero. The condition $< \sqrt{\frac{4L}{C}}$.

The angular frequency of oscillations is given by $\omega = \frac{2\pi}{T} = 2\pi f$ where f is the frequency of oscillations. Thus $f = \frac{\omega}{2\pi} = \frac{\sqrt{(k^2 - b^2)}}{2\pi}$ as $\omega = \sqrt{(k^2 - b^2)}$

As $k^2 = \frac{1}{LC}$ and $b = \frac{R}{2L}$, the frequency of damped oscillations as $\boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$

If R is negligibly small then $\boldsymbol{f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}}$ **.** The graph above shows the variation of charge with time for different conditions.

Case 2. $\boldsymbol{b^2 = k^2}$ **or** $\boldsymbol{\frac{R^2}{4L^2} = \frac{1}{LC}}$ $\boldsymbol{or\ R = \sqrt{\frac{4L}{C}}}$ (**critically damped condition**), equation (10) becomes

$q = \frac{1}{2} q_0 e^{-bt}$. In this case the charge decays exponentially to zero.

Case 3. $\boldsymbol{b^2 > k^2}$ **or** $\boldsymbol{\frac{R^2}{4L^2} > \frac{1}{LC}}$ $\boldsymbol{or\ R > \sqrt{\frac{4L}{C}}}$ (**over damped condition**), equation (10) is expressed as it is. i.e.

$$q = \frac{1}{2} q_0 e^{-bt} \left\{ \left(1 + \frac{b}{\sqrt{b^2 - k^2}}\right) e^{(\sqrt{b^2 - k^2})t} + \left(1 - \frac{b}{\sqrt{b^2 - k^2}}\right) e^{-(\sqrt{b^2 - k^2})t} \right\}$$ Here the charge decays slowly and takes a long time to reach zero.

Note : In a LCR series circuit, as the charge oscillates, its amplitude of oscillation decreases due to dissipation energy across resistor as thermal energy. In an LCR series circuit, the electromagnetic energy is due to electric energy stored in a capacitor and magnetic energy stored in an inductor. There is continuous variation of current, charge and voltage across the components responsible for this energy.

The total electromagnetic energy at any instant of time t when the current is I and charge is q is $U = U_E + U_B = \frac{q^2}{2C} + \frac{1}{2}LI^2$ where U_E is the energy stored in a capacitor and U_B is the energy stored in an inductor.

The rate of transfer of thermal energy across the resistor is $\frac{dU}{dt} = -IR^2$. The negative sign indicates that the energy is continuously decreasing. In the absence of resistance, the total energy remains constant oscillating between the electric and magnetic energies across capacitor and inductor respectively. This is similar to an oscillating pendulum where energy is continuously changing between potential and kinetic energies.

PART-A

1 Define self-inductance of a given coil. Arrive at the expression for the energy stored in an inductor.

2 Arrive at the expression for the self inductance of a solenoid in terms of number of turns and area of the solenoid.

3 Obtain an expression for the growth of current in an **LR** circuit with a battery of emf E connected in series with **L** and **R.** Sketch the variation of current with time. Mark the inductive time constant on the graph.

4 Derive an expression for the decay of current in an **LR** circuit after the source battery of **E** is connected sufficiently long for the current to be maximum, i.e. $\mathbf{i_0}$ and then the battery is removed. What is the value of $\mathbf{i_0}$?

5 Obtain an expression for the growth of charge in a **RC** circuit with an emf **E** in series with **R** and **C**. Define capacitive time constant. Represent variation graphically.

6 Obtain an expression for charge in a series **R-C** circuit after the **C** has been Charged to $\mathbf{q_0}$ and then the battery has been removed. Define capacitive time constant.

7 Write a short note on the energy stored in the **LC** circuit. Explain in terms of the electric energy, magnetic energy and the total energy.

8. Derive an expression for the decay of charge of a capacitor in an LCR parallel circuit.

9. Explain the conditions of overdamping, critical damping and underdamping in a LCR series circuit.

PART-B

1 The magnetic flux passing through a coil of 200 turns is 0.5 mwb, when a 2 A current passes through it. Find its inductance. $\left[Hint : L = \frac{N\varphi}{I}\right]$

2 The self inductance of a coil is 3 mH and an electric current of 5 A is flowing through it. On switching off the current from 5 A to zero in 0.1 s. Calculate the average self induced emf in the coil. $\left[Hint : e = \left|L\frac{dI}{dt}\right|\right]$

3 A coil having a resistance of 15 Ω and an inductance of 10 H is connected to a 90 V battery. Determine the value of current (i) after 0.67 s and after (ii) 2s.

$\left[Hint: I_0 = \frac{E}{R},\ \tau = \frac{L}{R}\ and\ I = I_0\left(1 - e^{-\frac{R}{L}t}\right)\right]$

4 A sensitive electronic device of resistance 175 Ω is designed to operate with a current of 36 mA. To avoid damage to the device, an inductor is connected in series so that the current can rise to 4.9 mA in the first 58 μs. (a) What emf must the source have? (b) What inductance is required to satisfy the given condition? (c) What is the time constant? $\left[Hint: (a)\ I_0 = 36 \times 10^{-3} = \frac{E}{R},\ (b)\ I = 4.9 \times 10^{-3} = I_0\left(1 - e^{-\frac{R}{L}t}\right), Find\ L\ (c)\tau = \frac{L}{R}\right]$

5 A Coil has an inductance of 5.3 mH and a resistance of 0.35Ω. (a) If a 12 V emf is applied across the coil, how much energy is stored in the magnetic field after the current has built up its equilibrium value? (b) After how many time constants will half this equilibrium energy be stored in the magnetic field?

$\left[Hint: I_0 = \frac{E}{R},\ U = \frac{1}{2}LI_0^2,\ For\ U' = \frac{U}{2},\ I = I_0\left(1 - e^{-\frac{t}{\tau}}\right)\right]$

6 A capacitor of 1μF is first charged and then discharged through a resistance of 1MΩ. Calculate the time in which the charge on the capacitor will fall to 36.8% of its initial value. $\left[Hint: q = 0.368q_0,\ q = q_0e^{-\frac{t}{RC}}\right]$

7 A capacitor of capacity 0.5 μF is discharged through a resistance of 10 MΩ. Find the time taken for half the charge on the capacitor to escape. $\left[Hint: q = \frac{1}{2}q_0,\ q = q_0e^{-\frac{t}{RC}}\right]$

8 In an oscillatory LCR circuit L = 0.2 H, C = 0.0012 μF.What is the maximum value of resistance for the circuit to be oscillatory? $\left[Hint : R^2 = \frac{4L}{c}\right]$

9 An LCR circuit has L = 0.45H, C = 2.5× 10^{-5} F and resistance R. (a) what is the frequency of the circuit when R = 0? Also calculate the angular frequency. What

value must R have to give a 5% decreases in angular frequency compared to the value calculated in the previous case?

$\left[Hint: f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\ ,\ \ \omega = \sqrt{\frac{1}{LC}},\ \omega' = 95\%\ of\ \omega,\ \ to\ find\ R\ use\ \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]$

10 An oscillating LC circuit consisting of a 1.0 nF capacitor and a 3.0 mH coil has a maximum voltages of 3.0 V. (a) What is the maximum charge in the capacitor? (b) What is the maximum current through the circuit? (c) What is the maximum energy stored in the magnetic field of the coil? $\left[Hint: (a)\ q_0 = EC,\ \ (b) I_0 = I_0 = \omega q_0,\ U = \frac{1}{2}LI_0^2\right]$

11 A capacitor of capacitance 1μF is charged to 200V. If it is discharged through a resistance of 200Ω in what time its potential drop to 100V? $\left[Hint: q_0 = EC,\ \ charge\ corresponding\ to\ V' = 100V\ is\ q' = E'C,\ q' = q_0 e^{-\frac{t}{RC}}\right]$

12 A capacitor discharges through an inductor of 0.1H and a resistor of 100Ω. If the frequency of oscillation is 1000 Hz, calculate the capacitance. $\left[Hint: f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]$

13 Calculate the value of the current after 1 second when a p.d of 10V is applied to a circuit of resistance 20 ohm and self-inductance 0.1H in series.

$\left[Hint: I_0 = \frac{E}{R},\ \ and\ I = I_0\left(1 - e^{-\frac{R}{L}t}\right)\right]$

14 A capacitor of capacitance 1 μF is discharged through a resistance. Time taken for half the charge on the capacitor to leak is found to be 10 s. Calculate the value of resistance. $\left[Hint: q = \frac{1}{2}q_0\ ,\ \ q = q_0 e^{-\frac{1}{RC}t}\right]$

15 A capacitor of capacitance 1 μF is connected to a battery of 2 V through a resistance of 10 kΩ. Calculate the initial current and current after 0.02 s.

$\left[Hint: q = q_0\left(1 - e^{-\frac{1}{RC}t}\right) dividing\ by\ C\ on\ either\ side\ and\ putting\ \frac{q}{C} = V\ and\ \frac{q_0}{C} = V_0, we\ have\ V = V_0\left(1 - e^{-\frac{1}{RC}t}\right)\ Use\ I = \frac{V}{R}\ \ and\ I_0 = \frac{V_0}{R}\right]$

16 A potential difference of 1 V is applied to a coil of resistance 2Ω and self inductance 2 H. What is the current after 0.1 s? $\left[Hint: I = I_0\left(1 - e^{-\frac{R}{L}t}\right)\right]$

17 A capacitor of capacitance 1 μF is charged to 200 V. If it is discharged through a resistance of 200 Ω In what time will its potential drop to 100V?

$\left[Hint: \frac{q}{C} = V\ and\ \frac{q_0}{C} = V_0, we\ have\ V = V_0\left(1 - e^{-\frac{1}{RC}t}\right)\right]$

18 In an LR circuit, the current attains (1/3) of its final steady value in one second after the circuit is closed. What is the time constant of the circuit?

$\left[Hint: I = \frac{1}{3}I_0,\ I = I_0\left(1 - e^{-\frac{R}{L}t}\right)\ \tau = \frac{L}{R}\right]$

19 A coil having an inductance of 20 H and a resistance of 10 Ω is connected to a battery. How long will it take for the current to reach half of the final value?

$\left[Hint: I = \frac{1}{2}I_0,\ I = I_0\left(1 - e^{-\frac{R}{L}t}\right)\right]$

PART C

1. The inductance coils are made of copper. Why?

Ans: As copper has small resistance, large induced currents can be produced by the copper coils.

2. A coil is removed from the magnetic field (i) rapidly (ii) slowly. In which case more work will be done?

Ans: When the coil is moved swiftly, more work is done and a larger induced emf is produced.

3. Is there any loss of energy due to the production of back emf in an LR circuit? Why?

Ans: No. During the growth of current, the energy is stored in the coil as magnetic field. This energy is used during the decay.

4. What is the significance of time constant of an R-L series circuit?

Ans: Time constant is a measure of the rate at which current grows or decays in a LR circuit.

5. Can we connect a capacitor directly to a dc source? Explain.

No. The resistance is negligible and cannot limit the current and the capacitor can act as a short circuit. A resistance need to be connected along with capacitor across a source of voltage so that capacitor does not get damaged.

6 Self inductance of a coil is also called electrical inertia. Explain

Ans: It is called the electrical inertia similar to mass which is referred to as measure of inertia in a mechanical system. If mass is large then more is the opposition to motion, similarly greater is the inductance, larger is the back emf which opposes the growth of current.

7 When does LCR circuit get critically damped?

Ans: For a resistance equal to $R = \sqrt{\frac{4L}{C}}$ the LCR circuit is said to be critically damped where the variation is exponential.

8 How do you increase the time constant of CR circuit?

Ans: The time constant of a CR circuit is given by $\tau = CR$. Thus by either increasing the value of capacitance or resistance or both, it is possible to increase the time constant.

9 What happens to the frequency of oscillation in an LCR circuit when the capacitor is filled with a dielectric?

Ans: The frequency of the LCR series circuit is given by $f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$. When a capacitor is filled with a dielectric medium, its capacitance increases and hence frequency decreases.

10 How do you increase the time constant of LR circuit?

Ans: The time constant of a LR circuit is given by $\tau = \frac{L}{R}$. Thus by either increasing the value of inductance or by decreasing the value of resistance, it is possible to increase the time constant.

11 Is total energy conserved in a LC circuit with negligible resistance? Explain.

Ans: Energy is transferred between the electric fields in a capacitor and magnetic fields in a inductor. When the capacitor is completely charged energy is purely electric energy in a capacitor for zero current. When the current is maximum the charge on the capacitor is zero and energy is completely magnetic energy in a inductor. At any other time it is partially electric and partially magnetic but total energy remains constant.

12 How do you increase the frequency of oscillations in a LC circuit?

Ans: The frequency of oscillations is given by $f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$. *Either by decreasing the inductance or capacitance, the frequency can be increases. The amplitude of the oscillations remain constant.*